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Some geometric properties of Read's space ☆

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A B S T R A C T

We study geometric properties of the Banach space $\mathcal{R}$ constructed recently by C. Read [8] which does not contain proximinal subspaces of finite codimension greater than or equal to two. Concretely, we show that the bidual of $\mathcal{R}$ is strictly convex, that the norm of the dual of $\mathcal{R}$ is rough, and that $\mathcal{R}$ is weakly locally uniformly rotund (but it is not locally uniformly rotund). Apart of the own interest of the results, they provide a simplification of the proof by M. Rmoutil [9] that the set of norm-attaining functionals over $\mathcal{R}$ does not contain any linear subspace of dimension greater than or equal to two. Note that if a Banach space X contains proximinal subspaces of finite codimension at least two, then the set of norm-attaining functionals over X contain two-dimensional linear subspaces of X^*. Our results also provides positive answer to the questions of whether the dual of $\mathcal{R}$ is smooth and of whether $\mathcal{R}$ is weakly locally uniformly rotund [9]. Finally, we present a renorming of Read's space which is smooth, whose dual is smooth, and such that its set of norm-attaining functionals does not contain any linear subspace of dimension greater than or equal to two, so the renormed space

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does not contain proximinal subspaces of finite codimension greater than or equal to two.

1. Introduction

In his recent brilliant manuscript [8], the late Charles J. Read constructed an equivalent norm $|||\cdot|||$ on c_0 such that the space $\mathcal{R} = (c_0, |||\cdot|||)$ answers negatively the following open problem by Ivan Singer from 1974 [10]:

(S) Is it true that every Banach space contains a proximinal subspace of finite codimension greater than or equal to 2?

Recall that a subset Y of a Banach space X is said to be *proximinal* if for every $x \in X$ there is $y_0 \in Y$ such that $\|x - y_0\| = \inf\{\|x - y\| \colon y \in Y\}$.

Many times, when an interesting and non-trivial space is constructed to produce a counterexample, it can be also used to solve some other different problems. In the case of Read's space, this didn't wait for too long: in [9, Theorem 4.2], Martin Rmoutil demonstrates that the space $\mathcal{R}$ also gives a negative solution to the following problem by Gilles Godefroy [5, Problem III]:

(G) is it true that for every Banach space X the set $\mathrm{NA}(X) \subset X^*$ of norm attaining functionals contains a two-dimensional linear subspace?

Recall that an element f of the dual X^* of a Banach space X is said to be *norm attaining* if there is $x \in X$ with $\|x\| = 1$ such that $\|f\| = |f(x)|$.

The utility of Read's space makes clear that it is interesting to increase our knowledge of its geometry. The aim of this note is to show that Read's space fulfills the following properties:

(a) the bidual $\mathcal{R}^{**}$ of Read's space is strictly convex;
(b) therefore, $\mathcal{R}^*$ is smooth; and
(c) $\mathcal{R}$ is also strictly convex;
(d) moreover, $\mathcal{R}$ is weakly locally uniformly rotund (WLUR);
(e) the norm of $\mathcal{R}^*$ is rough, so it is not Fréchet differentiable at any point;

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