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Green's function asymptotics of periodic elliptic operators on abelian coverings of compact manifolds

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ABSTRACT

The main results of this article provide asymptotics at infinity of the Green's functions near and at the spectral gap edges for "generic" periodic second-order, self-adjoint, elliptic operators on noncompact Riemannian co-compact coverings with abelian deck groups. Previously, analogous results have been known for the case of $\mathbb{R}^n$ only. One of the interesting features discovered is that the rank of the deck group plays more important role than the dimension of the manifold.

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1. Introduction

The behavior at infinity of the Green's function of the Laplacian $-\Delta$ on an Euclidean space below and at the boundary of the spectrum is well known. The main term of the asymptotics for the Green's function of any bounded below periodic second-order elliptic operator below and at the bottom of the spectrum was found in [7,31] (see also [44] for discrete setting). For such operators, the band-gap structure of their spectra is known (e.g., [12,37]), and thus, spectral gaps may exist. Thus, it is interesting to derive the behavior of the Green's functions inside and at the edges of these gaps. Recently, the corresponding results for a "generic" periodic elliptic operator in $\mathbb{R}^n$ were established in [19,27].

Meanwhile, many classical properties of solutions of periodic Schrödinger operators on Euclidean spaces were generalized successfully to solutions of periodic Schrödinger operators on coverings of compact manifolds (see e.g., [3,8,9,22,26,41,42,28]). Hence, a question arises of whether one can obtain analogs of the results of [19,27] as well. The main Theorems 3.1 and 3.4 of this article provide such results for periodic operators on an abelian covering of a compact Riemannian manifold. The results are in line with Gromov's idea that the large scale geometry of a co-compact normal covering is captured mostly by its deck transformation group (see e.g., [10,14,39]). For instance, the dimension of the covering manifold does not enter explicitly to the asymptotics. Rather, the torsion-free rank d of the abelian deck transformation group influences these asymptotics significantly. One can find a similar effect in various results involving analysis on Riemannian co-compact normal coverings such as the long time asymptotic behaviors of the heat kernel on a noncompact abelian Riemannian covering [23], and the analogs of Liouville's theorem [26] (see also [39] for an excellent survey on analysis on co-compact coverings).

We discuss now the main thrust of this paper.

Let X be a noncompact Riemannian manifold that is a normal abelian covering of a compact Riemannian manifold M with the deck transformation group G. For any function u on X and any $g \in G$, we denote by u^g the "shifted" function

$$u^g(x) = u(g \cdot x),$$

for any $x \in X$. Consider a bounded below second-order, symmetric elliptic operator L on the manifold X with smooth coefficients. We assume that L is a **periodic** operator on X, i.e., the following invariance condition holds:

$$Lu^g = (Lu)^g,$$

for any $g \in G$ and $u \in C_c^\infty(X)$. The operator L, with the Sobolev space $H^2(X)$ as its domain, is an unbounded self-adjoint operator in $L^2(X)$.

The following result for such operators is well-known (see e.g., [8,12,24,37,41,42]):

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