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Global well-posedness and blow-up for the 2-D Patlak–Keller–Segel equation

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ABSTRACT

In this paper, we prove that for every nonnegative initial data in $L^1(\mathbb{R}^2)$, the Patlak–Keller–Segel equation is globally well-posed if and only if the total mass $M \leq 8\pi$. Our proof is based on some monotonicity formulas of nonnegative mild solutions.

1. Introduction

In this paper, we consider the parabolic–elliptic Patlak–Keller–Segel model in $\mathbb{R}^2$:

$$\begin{cases} n_t - \triangle n + \mathrm{div}(n\nabla c) = 0, \\ -\triangle c = n, \\ n|_{t=0} = n_0(x). \end{cases} \tag{1.1}$$

Here $n(x,t)$ represents the cell density, and $c(x,t)$ is the concentration of chemo-attractant. The Patlak–Keller–Segel system describes the collective motion of cells which

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are attracted by a chemical substance and are able to emit it. For the background of this system, one can refer to [22,24] and review papers [19,20].

We say that $n \in C_w([0,T), L^1(\mathbb{R}^2))$ is a weak solution to (1.1) if for all test functions $\psi \in \mathcal{D}(\mathbb{R}^2)$,

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi(x)n(x,t)dx = \int_{\mathbb{R}^2} \triangle\psi(x)n(x,t)dx$$
$$-\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [\nabla\psi(x) - \nabla\psi(y)] \cdot \frac{x-y}{|x-y|^2} n(x,t)n(y,t)dx\, dy. \tag{1.2}$$

Here $C_w([0,T), L^1(\mathbb{R}^2))$ is a subspace of $L^\infty([0,T), L^1(\mathbb{R}^2))$ such that $t \mapsto \int_{\mathbb{R}^2} \psi(x) \times n(x,t)dx$ is continuous for any $\psi \in \mathcal{D}(\mathbb{R}^2)$. The notion of weak solution was introduced in [25]. Free energy solutions are nonnegative weak solutions, which satisfy the integrability condition $(1 + |x|^2 + |\ln n|)n \in L^\infty([0,T), L^1(\mathbb{R}^2))$ and the free energy inequality

$$F[n(0)] \geq F[n(t)] + \int_0^t \int_{\mathbb{R}^2} n(x,s)|\nabla \ln n(x,s) - \nabla c(x,s)|^2 dxds,$$

for *a.e.* $t \in (0,T)$, where the free energy functional $F(n)$ is given by

$$F[n] = \int_{\mathbb{R}^2} n \ln n dx - \frac{1}{2}\int_{\mathbb{R}^2} ncdx, \quad c(x,s) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|n(y,s)dy.$$

The free energy functional was introduced for chemotactic models by Nagai, Senba and Yoshida in [23], by Biler in [3], and by Gajewski and Zacharias in [16]. The concept of free energy solutions is necessary to apply the entropy method to analyze the asymptotic behavior of solutions.

Blanchet, Dolbeault and Perthame [7] proved the free energy inequality and found that there exists a critical mass $M_c = 8\pi$ so that the free energy solutions globally exist for the data $n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)dx)$ and $n_0 \log n_0 \in L^1(\mathbb{R}^2)$ with the mass $M < 8\pi$, which was announced in [13]. Here $M = \|n_0\|_{L^1}$ is the total mass of the initial data n_0. The proof is based on the free energy inequality and the logarithmic Hardy–Littlewood–Sobolev inequality. One can refer to [11] for an alternative proof using the idea from [12]. The case of critical mass $M = 8\pi$ was proved by Blanchet, Carrillo and Masmoudi [6] (see [4] for the radially symmetric solution). For the case of super-critical mass $M > 8\pi$, the solution will blow up in finite time, due to

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 n(x,t)dx = 4M\left(1 - \frac{M}{8\pi}\right), \tag{1.3}$$

see [14,7] for example. Recently, Fernández and Mischler [15] proved the uniqueness of the free energy solutions. Let us refer to [5,1,10,17] for more recent results.