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Modified scattering for the critical nonlinear Schrödinger equation

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ABSTRACT

We consider the nonlinear Schrödinger equation

$$iu_t + \Delta u = \lambda |u|^{\frac{2}{N}} u$$

in all dimensions $N \geq 1$, where $\lambda \in \mathbb{C}$ and $\Im\lambda \leq 0$. We construct a class of initial values for which the corresponding solution is global and decays as $t \to \infty$, like $t^{-\frac{N}{2}}$ if $\Im\lambda = 0$ and like $(t \log t)^{-\frac{N}{2}}$ if $\Im\lambda < 0$. Moreover, we give an asymptotic expansion of those solutions as $t \to \infty$. We construct solutions that do not vanish, so as to avoid any issue related to the lack of regularity of the nonlinearity at $u = 0$. To study the asymptotic behavior, we apply the pseudo-conformal transformation and estimate the solutions by allowing a certain growth of the Sobolev norms which depends on the order of regularity through a cascade of exponents.

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1. Introduction

In this article, we consider the nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda |u|^\alpha u \\ u(0,x) = u_0 \end{cases} \tag{1.1}$$

on $\mathbb{R}^N$, where

$$\alpha = \frac{2}{N} \tag{1.2}$$

and

$$\Im \lambda \le 0 \tag{1.3}$$

and its equivalent integral formulation

$$u(t) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u|^\alpha u \, ds \tag{1.4}$$

where $(e^{it\Delta})_{t\in\mathbb{R}}$ is the Schrödinger group.

It is well known that the Cauchy problem for (1.1)–(1.3) is globally well posed in a variety of spaces, for instance in $H^1(\mathbb{R}^N)$, in $L^2(\mathbb{R}^N)$, and in

$$\Sigma = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx). \tag{1.5}$$

See e.g. [14]. Concerning the long time asymptotic behavior of the solutions, $\alpha = \frac{2}{N}$ is a limiting case. Indeed, for $\alpha > \frac{2}{N}$, there is low energy scattering, i.e. a solution of (1.1) with a sufficiently small initial value (in some appropriate sense) is asymptotic as $t \to \infty$ to a solution of the free Schrödinger equation. See [21,8,7,5,6,16,4]. On the other hand, if $\alpha \le \frac{2}{N}$, then low energy scattering cannot be expected, see [20, Theorem 3.2 and Example 3.3, p. 68] and [1].

In the case $\alpha = \frac{2}{N}$, the relevant notion is modified scattering, i.e. standard scattering modulated by a phase. When $\Im \lambda = 0$, the existence of modified wave operators was established in [17] in dimension $N = 1$. More precisely, for all sufficiently small asymptotic state u^+, there exists a solution of (1.1) which behaves as $t \to \infty$ like $e^{i\phi(t,\cdot)} e^{t\Delta} u^+$, where the phase ϕ is given explicitly in terms of u^+. (See also [2]. See [12,19] for extensions in dimension $N = 2$.) Conversely, for small initial values, it was proved in [9] that the asymptotic behavior of the corresponding solution has this form when $\Im \lambda = 0$, in dimensions $N = 1, 2, 3$. (See also [15].) If $\Im \lambda < 0$, then the nonlinearity has some dissipative effect, and an extra log decay appears in the description of the asymptotic