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Superconcentration, and randomized Dvoretzky's theorem for spaces with 1-unconditional bases

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Let n be a sufficiently large natural number and let B be an origin-symmetric convex body in $\mathbb{R}^n$ *in the ℓ-position*, and such that the space $(\mathbb{R}^n, \|\cdot\|_B)$ admits a 1-unconditional basis. Then for any $\varepsilon \in (0, 1/2]$, and for random $c\varepsilon \log n/\log\frac{1}{\varepsilon}$-dimensional subspace E distributed according to the rotation-invariant (Haar) measure, the section $B\cap E$ is $(1+\varepsilon)$-Euclidean with probability close to one. This shows that the "worst-case" dependence on ε in the randomized Dvoretzky theorem in the ℓ-position is significantly better than in John's position. It is a previously unexplored feature, which has strong connections with the concept of superconcentration introduced by S. Chatterjee. In fact, our main result follows from the next theorem: Let B be as before and assume additionally that B has a smooth boundary and $\mathbb{E}_{\gamma_n}\|\cdot\|_B \le n^c\,\mathbb{E}_{\gamma_n}\big\|\mathrm{grad}_B(\cdot)\big\|_2$ for a small universal constant $c > 0$, where $\mathrm{grad}_B(\cdot)$ is the gradient of $\|\cdot\|_B$ and γ_n is the standard Gaussian measure in $\mathbb{R}^n$. Then for any $p \in [1, c\log n]$ the p-th power of the norm $\|\cdot\|_B^p$ is $\frac{C}{\log n}$-superconcentrated in the Gauss space.

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1. Introduction

The term *superconcentration* was introduced by S. Chatterjee to describe a situation when the size of typical fluctuations of a function on a probability space is much smaller than the bound provided by "classical" concentration inequalities [5]. In this note, we are concerned with applications of the superconcentration phenomenon in asymptotic geometric analysis; specifically, in the problem of finding large $(1+\varepsilon)$-Euclidean sections of convex bodies. On the probabilistic level, we derive a concentration inequality for convex positively homogeneous functions in the Gauss space satisfying some additional assumptions. On the geometric level, we show that John's position may be a "bad" choice as far as dependence of dimension on ε is concerned in the *randomized* Dvoretzky's theorem, and, at least for unit balls of normed spaces with a 1-unconditional basis, the *ℓ-position* allows a substantially better bound on the dimension.

The theorem of A. Dvoretzky [9] asserts that for arbitrary fixed $k \in \mathbb{N}$ and $\varepsilon > 0$, every origin-symmetric convex body of a large enough dimension contains a $(1+\varepsilon)$-Euclidean k-dimensional section. A proof of the theorem based on the concentration of measure was proposed by V. Milman [18]. In view of results of Y. Gordon [12] and G. Schechtman [24], who improved the dependence of the dimension k on ε, the theorem of V. Milman reads: If B is an origin-symmetric convex body in $\mathbb{R}^n$ with the Minkowski functional $\|\cdot\|_B$ and

$$k(B) := \left(\frac{\mathbb{E}\|G\|_B}{\mathrm{Lip}(\|\cdot\|_B)}\right)^2$$

(where G is the standard Gaussian vector in $\mathbb{R}^n$ and $\mathrm{Lip}(\|\cdot\|_B)$ is the Lipschitz constant of $\|\cdot\|_B$), then for any $\varepsilon \in (0,1]$ and any natural $k \le c\varepsilon^2 k(B)$ the random k-dimensional subspace $E \subset \mathbb{R}^n$ uniformly distributed according to the rotation-invariant measure, cuts a $(1+\varepsilon)$-Euclidean section $B \cap E$ with probability close to one. The quantity $k(B)$ is often called the critical, or Dvoretzky's, dimension. The last statement asserts that *most* sections of B (with respect to the rotation-invariant probability measure) of the given dimension are $(1+\varepsilon)$-Euclidean; in our note this version of Dvoretzky's theorem is called "randomized" (as opposed to "existential"). Let us note that Dvoretzky's theorem as well as numerous questions around it are covered in several monographs and surveys; see, in particular, [19,23,25,2].

The Dvoretzky–Rogers lemma implies that for any convex body B in *John's position* (i.e. such that the ellipsoid of maximal volume contained inside B is the unit Euclidean ball) one has $\mathbb{E}\|G\|_B \ge c\sqrt{\log n}$, whence $k(B) \ge c^2 \log n$ for a universal constant $c > 0$. This yields

Theorem 1 *(Randomized Dvoretzky's theorem in John's position, [18,12,24]). Let B be an origin-symmetric convex body in $\mathbb{R}^n$ in John's position, and let $\varepsilon \in (0,1]$ and $k \le c'\varepsilon^2 \log n$. Then, for random k-dimensional subspace E uniformly distributed according to the rotation-invariant (Haar) measure, one has*

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