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# Vanishing of hyperelliptic L-functions at the central point

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ABSTRACT

We obtain a lower bound on the number of quadratic Dirichlet L-functions over the rational function field which vanish at the central point $s = 1/2$. This is in contrast with the situation over the rational numbers, where a conjecture of Chowla predicts there should be no such L-functions. The approach is based on the observation that vanishing at the central point can be interpreted geometrically, as the existence of a map to a fixed abelian variety from the hyperelliptic curve associated to the character.



## 1. Introduction

S. Chowla conjectured in [3] that, for any real non-principal Dirichlet character $\chi$, $L(s, \chi) \neq 0$ for all $s \in (0, 1)$. In particular, his conjecture asserts that L-functions of quadratic characters never vanish at the central point $s = 1/2$.

Although this conjecture is still open, much progress has been made. K. Soundararajan [14] proved that at least 87.5% of odd squarefree positive integers $d$ have the property $L(1/2, \chi_{8d}) \neq 0$ where $\chi_{8d}$ denotes the quadratic character with conductor $8d$.

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In this paper, we consider the analogue of Chowla's conjecture obtained by replacing the field of rational numbers with the field of rational functions over a finite field.

Let $q = p^e$ be a power of an odd prime $p$ and $\mathbb{F}_q$ the finite field with $q$ elements. Let $k = \mathbb{F}_q(t)$ denote the field of rational functions over $\mathbb{F}_q$. The primes of $k$ are represented by monic irreducible polynomials in $\mathbb{F}_q[t]$ except the one prime at infinity.

A quadratic character of $k$ corresponds to a squarefree polynomial in $\mathbb{F}_q[t]$. Explicitly, take $D \in \mathbb{F}_q[t]$ to be a squarefree polynomial and $K = k(\sqrt{D})$ the quadratic extension of $k$ by joining $\sqrt{D}$. Then we can define a quadratic character $\chi_D$ as follows:

For $P$ a prime of $k$,

$$\chi_D(P) = \begin{cases} 1 & P \text{ splits in } K \\ -1 & P \text{ inert in } K \\ 0 & P \text{ ramifies in } K \end{cases}$$

We define the L-function associated to $\chi_D$ as

$$L(s, \chi_D) = \prod_P (1 - \chi_D(P)|P|^{-s})^{-1}$$

where the product is taken over the primes represented by polynomials $P$ and $|P| = q^{degP}$.

**Definition 1.1.** Define sets:

$$P(N) = \{D \in \mathbb{F}_q[t] : D \text{ monic, squarefree, } |D| < N\}$$
$$g(N) = \{D \in P(N) : L(1/2, \chi_D) = 0\}.$$

**Remark 1.2.** Note that in the definition above, we have restricted ourselves to characters corresponding to monic squarefree polynomials which is half of all quadratic characters. But since we only study the density in this paper, such restriction won't affect our results.

Under this definition, the analogue of Chowla's conjecture states that $g(N)$ is empty for any $N$. There are some results towards this statement.

Bui and Florea [2] showed for a fixed finite field $\mathbb{F}_q$ with odd characteristic, as $N \to \infty$,

$$|g(N)| \leq 0.057N + o(1)$$

where $N = q^{2n+1}$ for some $n > 0$.

The purpose of this paper is to show that the analogue of Chowla's conjecture over $\mathbb{F}_q(t)$ is not correct and give a lower bound on the number of counterexamples with bounded height.