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The reciprocal sum of primitive nondeficient numbers

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ABSTRACT

We investigate the reciprocal sum of *primitive nondeficient numbers*, or pnds. In 1934, Erdős showed that the reciprocal sum of pnds converges, which he used to prove that the abundant numbers have a natural density. We show the reciprocal sum of pnds is between 0.348 and 0.380.

1. Introduction

The field of probabilistic number theory got its start in the 1920s and 1930s with work of Schoenberg [22], Davenport [5], and others who proved the existence of distribution functions for $\varphi(n)/n$, $\sigma(n)/n$, and similar functions. (Here, φ is Euler's function and σ is the sum-of-divisors function.) This line of work led up to the celebrated theorems of

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Erdős–Wintner and Erdős–Kac. The present paper is concerned with one of the earlier results in this history, namely Erdős [7], where an elementary argument is presented to show that the density of the set of n with $\sigma(n)/n \geq 2$ exists.

With nomenclature going back to ancient times, a number n is said to be abundant, deficient, or perfect if $\sigma(n)/n$ is greater than, less than, or equal to 2, respectively. One then says n is nondeficient if $\sigma(n)/n \geq 2$. For such n, since $\sigma(n)/n = \sum_{d|n} 1/d$, all multiples of n are also nondeficient. This naturally leads one to consider nondeficient numbers all of whose proper divisors are deficient, so-called *primitive nondeficient* (pnd) numbers. The sequence of pnds is OEIS A281505.

It is easy to see that every nondeficient number has a pnd divisor. The proof of Erdős [7] then hinged on showing that the reciprocal sum of the pnds is convergent. The convergence was shown by determining a sufficiently small upper bound on the counting function for pnds. Denoting the number of pnds $\leq x$ by $N(x)$, the paper showed that

$$N(x) = o\left(\frac{x}{(\log x)^2}\right),$$

which is enough to prove that the sum of reciprocals of the pnds converges. A more detailed study by Erdős in [8] found that, for sufficiently large x,

$$x/\exp(c_1\sqrt{\log x \log\log x}) \leq N(x) \leq x/\exp(c_2\sqrt{\log x \log\log x}),$$

where $c_1 = 8$ and $c_2 = 1/25$. Later, Ivić [11] improved the above bounds with $c_1 = \sqrt{6}+\epsilon$ and $c_2 = 1/\sqrt{12} - \epsilon$ for any fixed $\epsilon > 0$. Presumably there is a constant c such that $c_1, c_2 = c + o(1)$ as $x \to \infty$. Recent numerical experiments of Silva [23] suggest such a c may be close to 1. The best that is now known asymptotically is a result of Avidon [1] who showed we may take $c_1 = \sqrt{2} + \epsilon$ and $c_2 = 1 - \epsilon$.

Once a series is found to converge, it is natural to wonder what its value may be. For example, by Brun's Theorem it is known that the reciprocal sum of twin primes converges. This sum, called Brun's constant, is approximately 1.902160583104, which is found by extrapolating via the Hardy–Littlewood heuristics. However, the best proven upper bound is 2.347, see [4,12]. Similarly, Pomerance [19] proved that the reciprocal sum of numbers in amicable pairs converges, and work has also been done to determine bounds on this value, the Pomerance constant, the current bounds being 0.0119841556 and 215, see [2,18]. Given the existing nomenclature, we call the value of the reciprocal sum of pnds the *Erdős constant*.

The principal result of this paper is the following theorem.

Theorem 1.1. *The Erdős constant* $\sum_{n \text{ is a pnd}} 1/n$ *lies in the interval*

$$(0.34842\ ,\ 0.37937).$$

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