Journal of Number Theory 191 (2018) 289–304

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Bounds for the Petersson norms of the pullbacks of Saito–Kurokawa lifts

Pramath Anamby *, Soumya Das

Department of Mathematics, Indian Institute of Science, Bangalore – 560012, India

ARTICLE INFO

Article history:
Received 14 December 2017
Received in revised form 7 March 2018
Accepted 14 March 2018
Available online 17 April 2018
Communicated by F. Pellarin

MSC:
primary 11F11, 11F46
secondary 11F66

Keywords:
Pullbacks
Saito–Kurokawa lifts
Petersson norms
Mass distribution

ABSTRACT

Using the amplification technique, we prove that 'mass' of the pullback of the Saito–Kurokawa lift of a Hecke eigen form $g \in S_{2k}$ is bounded by $k^{1-\frac{1}{210}+\varepsilon}$. This improves the previously known bound k for this quantity.

1. Introduction

Let ℓ be an integer and M_ℓ^2 denote the space of Siegel modular forms of weight ℓ and degree 2 on $\mathrm{Sp}_2(\mathbf{Z}) (\subseteq M_4(\mathbf{Z}))$ and by S_ℓ^2 the subspace of cusp forms. These are holomorphic functions defined on the Siegel upperhalf space $\mathbf{H}_2$ which consists of complex symmetric matrices $Z \in M_4(\mathbf{C})$ whose imaginary part is positive-definite. If we

* Corresponding author.
E-mail addresses: pramatha@iisc.ac.in (P. Anamby), soumya@iisc.ac.in (S. Das).

https://doi.org/10.1016/j.jnt.2018.03.011
0022-314X/

write such a $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$, then $F|_{z=0} := F\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & \tau' \end{smallmatrix}\right)\right)$ is a modular form in τ and τ' (see [3] for more details) with weight ℓ, which we call the pullback of F to $\mathbf{H} \times \mathbf{H}$.

The study of pullbacks of automorphic forms has a rich history, see eg., [1], [6], [7], [12]. In the context of Siegel modular forms, there are conjectures of Ikeda [7] relating such pullabacks to central values of L-functions. As an example, the Gross–Prasad conjecture would relate pullbacks of Siegel cusp forms of degree 2 to central critical values of L-functions for $\mathrm{GSp}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$. Ichino's beautiful result [6] studies this question for the Saito–Kurokawa (SK from now on) lifts of elliptic modular forms. Following the above notation, let us write $F|_{z=0} = \sum_{g_1,g_2} c_{g_1,g_2} g_1(\tau) g_2(\tau')$ (see also [12]), where g_j runs over a Hecke basis of S_ℓ, the space of elliptic cusp forms on $\mathrm{SL}_2(\mathbf{Z})$. Then Ichino proves that if $F = F_g$ is the SK-lift of $g \in S_{2\ell-2}$ in the above, only the diagonal survives and $|c_{g_1,g_1}|$ is essentially given by the central value $L(1/2, \mathrm{sym}^2 g_1 \times g)$.

It was moreover observed in [12] that comparison of the (normalised) norm of F_g with the norm of its pullback provides a measure of the non-vanishing of the latter on average over the 'projection' of $F_g|_{z=0}$ along $g_1 \times g_1$, as $g_1 \in S_{2\ell-2}$ varies. By a formula (see (1.2)) in [12], this also provides a measure of density of F along $F|_{z=0}$ (see [12, (1.13)]). This is made more precise in the next paragraph. We now make a change of notation, and use $2k$ for the weight $2\ell - 2$, in conformity to the afore-mentioned papers on the topic.

For an odd integer $k > 0$, let $g \in S_{2k}$ be a normalized Hecke eigenform for $\mathrm{SL}_2(\mathbf{Z})$. Denote the Saito–Kurokawa lift of g by $F_g \in S^2_{k+1}$. Let us define the quantity

$$N(F_g) := \tfrac{1}{v_1^2} \left\langle F_g|_{z=0}, F_g|_{z=0} \right\rangle / \tfrac{1}{v_2} \left\langle F_g, F_g \right\rangle, \tag{1.1}$$

where $v_1 = \mathrm{vol.}(\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H})$ and $v_2 = \mathrm{vol.}(\mathrm{Sp}_2(\mathbf{Z})\backslash\mathbf{H}_2)$. Here $\langle F_g|_{z=0}, F_g|_{z=0}\rangle$ denotes the Petersson norm of $F_g|_{z=0}$ on $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H} \times \mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$ (see section 2 for more details).

Let B_{k+1} denote the Hecke basis for S_{k+1}. Now Ichino's formula [6] immediately implies the following identity which was shown in [12]:

$$N(F_g) = \frac{\pi^2}{15} \left(L(3/2, g) L(1, \mathrm{sym}^2 g)\right)^{-1} \cdot \frac{12}{k} \sum_{f \in B_{k+1}} L(\frac{1}{2}, \mathrm{sym}^2 f \times g). \tag{1.2}$$

It is this quantity $N(F_g)$ that we are concerned with in this paper and we refer this quantity as the *mass* of the pullback of the Saito–Kurokawa lift. Let us recall that as a special case of conjectures of [2], Liu and Young in [12] conjectured that $N(F_g) \sim 2$ as $k \to \infty$, and proved it on average over the family $g \in B_{2k}$ and $K \leq k \leq 2K$. In [1] a stronger asymptotic formula was obtained by considering only the smaller family $g \in B_{2k}$. Their result says that there exists some $\eta > 0$ such that

$$\frac{12}{2k-1} \sum_{g \in B_{2k}} N(F_g) = 2 + O(k^{-\eta}). \tag{1.3}$$