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The moments of the logarithm of a G.C.D. related to Lucas sequences

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ABSTRACT

Let $(u_n)_{n\geq 0}$ be a nondegenerate Lucas sequence satisfying $u_n = a_1 u_{n-1} + a_2 u_{n-2}$ for all integers $n \geq 2$, where a_1 and a_2 are some fixed relatively prime integers; and let g_u be the arithmetic function defined by $g_u(n) := \gcd(n, u_n)$, for all positive integers n. Distributional properties of g_u have been studied by several authors, also in the more general context where $(u_n)_{n\geq 0}$ is a linear recurrence. We prove that for each positive integer λ it holds

$$\sum_{n \,\leq\, x} (\log g_u(n))^\lambda \sim M_{u,\lambda} x$$

as $x \to +\infty$, where $M_{u,\lambda} > 0$ is a constant depending only on a_1, a_2, and λ. More precisely, we provide an error term for the previous asymptotic formula and we show that $M_{u,\lambda}$ can be written as an infinite series.

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1. Introduction

Let $(u_n)_{n\geq 0}$ be an integral linear recurrence, that is, $(u_n)_{n\geq 0}$ is a sequence of integers and there exist $a_1, \dots, a_k \in \mathbb{Z}$, with $a_k \neq 0$, such that

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k},$$

for all integers $n \geq k$. We recall that $(u_n)_{n\geq 0}$ is said to be *nondegenerate* if none of the ratios α_i/α_j $(i \neq j)$ is a root of unity, where $\alpha_1, \dots, \alpha_v \in \mathbb{C}$ are all the pairwise distinct roots of the *characteristic polynomial*

$$\psi_u(X) = X^k - a_1 X^{k-1} - a_2 X^{k-2} - \cdots - a_k.$$

Moreover, $(u_n)_{n\geq 0}$ is said to be a *Lucas sequence* if $u_0 = 0$, $u_1 = 1$, and $k = 2$. In particular, the Lucas sequence with $a_1 = a_2 = 1$ is known as the *Fibonacci sequence*. We refer the reader to [9, Chapter 1] for the basic terminology and theory of linear recurrences.

Let g_u be the arithmetic function defined by $g_u(n) := \gcd(n, u_n)$, for all positive integers n. Many researchers have studied the properties of g_u. For instance, the set of fixed points of g_u, that is, the set of positive integers n such that $n \mid u_n$, has been studied by Alba González, Luca, Pomerance, and Shparlinski [1], under the mild hypotheses that $(u_n)_{n\geq 0}$ is nondegenerate and that its characteristic polynomial has only simple roots; and by André-Jeannin [2], Luca and Tron [16], Sanna [21], and Somer [26], when $(u_n)_{n\geq 0}$ is a Lucas sequence or the Fibonacci sequence. This topic can be regarded as a generalization of the study of *Fermat pseudoprimes*. Indeed, when the linear recurrence is given by $u_n = a^{n-1} - 1$, for some fixed integer $a \geq 2$, then the composite integers $n \geq 2$ such that $g_u(n) = n$ are exactly the Fermat pseudoprimes to base a [8, Definition 9.9]. Also, it can be considered as the easiest nontrivial instance of the problem of studying when $v_n \mid u_n$ for "many" positive integers n, where $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ are fixed integral linear recurrences. This problem is due to Pisot and the major results have been given by van der Poorten [28], Corvaja and Zannier [6,7]. (See also [20] for a proof of the last remark in [7].) Furthermore, upper bounds for the generalization of g_u defined by $g_{u,v}(n) := \gcd(u_n, v_n)$, for all positive integers n, have been proved by Bugeaud, Corvaja, and Zannier [4], and by Fuchs [10], for large classes of linear recurrences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$.

On the other hand, Sanna and Tron [22,24] have investigated the fiber $g_u^{-1}(y)$, when $(u_n)_{n\geq 0}$ is nondegenerate and $y = 1$, and when $(u_n)_{n\geq 0}$ is the Fibonacci sequence and y is an arbitrary positive integer; while the image $g_u(\mathbb{N})$ have been studied by Leonetti and Sanna [14], in the case in which $(u_n)_{n\geq 0}$ is the Fibonacci sequence.

Moreover, fixed points and fibers of g_u have been studied also when $(u_n)_{n\geq 0}$ is an elliptic divisibility sequence [11,12,25], the orbit of 0 under a polynomial map [5], and the sequence of central binomial coefficients [17,23].

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