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Effective bounds for Fourier coefficients of certain weakly holomorphic modular forms

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ABSTRACT

In Jorgenson et al. (2016) [JST 16a], the authors derived generators for the function fields associated to certain low genus arithmetic surfaces realized through the action of the discrete Fuchsian group $\Gamma_0(N)^+/\{\pm 1\}$ on the upper half plane. In particular, they construct modular forms which are analogs to the modular discriminant and the Klein j-invariant of the full modular group $\mathrm{PSL}(2,\mathbb{Z})$. In this article, we produce effective and practical bounds for the Fourier coefficients in the q-expansion of such generators, thus allowing for rigorous numerical inspection of the generators.

1. Introduction

The action of the full modular group $\mathrm{PSL}(2,\mathbb{Z})$ on the upper-half plane $\mathbb{H}$ yields the quotient space $X_1 \cong \mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ which is a genus zero hyperbolic surface with one cusp equivalent to $i\infty$ and two elliptic fixed points equivalent to $z = i$ and $z = e^{i\pi/3}$ respectively. Functions defined on X_1 are usually referred to as modular forms and depending on their behavior at the cusp, such functions are either weakly holomorphic

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(if they have a pole at the cusp) or holomorphic. The latter ones are called cusp forms if they vanish at the cusp. The holomorphic Eisenstein series are the modular forms that generate the vector space of fixed even weight k holomorphic functions on X_1, namely functions satisfying $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma(z) = (az+b)/(cz+d) \in \mathrm{PSL}(2,\mathbb{Z})$. For even $k \geq 4$, the holomorphic Eisenstein series of weight k are given by

$$E_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{PSL}(2,\mathbb{Z})} (cz+d)^{-k}, \quad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \tag{1.1}$$

where $z \in \mathbb{H}$ and $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$. The smallest weight for which the vector space of holomorphic forms is non-trivial is 4. The Eisenstein series $E_4(z)$ and $E_6(z)$ are the generators for the graded ring of holomorphic modular forms. The smallest weight for which there exists a cusp form is 12. The modular discriminant $\Delta(z)$ given by

$$\Delta(z) = E_4^3(z) - E_6^2(z) = \eta(z)^{24} = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \quad q = e^{2\pi i z} \tag{1.2}$$

is the unique (up to multiplication by a constant) weight 12 cusp form.

The holomorphic Eisenstein series can be used to construct the field of rational functions (weakly holomorphic forms) on X_1. In particular, the field of automorphic forms (i.e. weight 0 weakly holomorphic) is generated by the Klein j-invariant which is a rational function in $E_4(z)$ and $E_6(z)$, namely

$$j(z) = \frac{12^3 E_4^3(z)}{\Delta(z)} \,. \tag{1.3}$$

The Fourier expansion of the j-invariant at the cusp $i\infty$ has the following form

$$j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + O(q^3) \text{ as } q \to 0 \,, \tag{1.4}$$

with the coefficients satisfying the bound of the form $c_n = O(n^{-3/4} e^{4\pi\sqrt{n}})$ as n approaches infinity (see [Ra 38]). There is a rather vast literature on the subject of the j-invariant, but for the sake of brevity of this article, we leave them out, referring the reader to [Ga 06].

In a series of recent articles ([JST 16a], [JST 16b], [JST 16c], [JST 18]), the authors consider analogs to the j-invariant for certain arithmetic subgroups $\Gamma_0(N)^+/\{\pm 1\}$, having square-free positive integer level N. Denote by $X_0(N)^+$ the arithmetic surface obtained via the action of the group on the upper-half plane $\mathbb{H}$. For the genus 0 such surfaces, there is only one generator $j_N(z)$ for the function field of automorphic forms. When the surface has genus g with $g \geq 1$, the function field on $X_0(N)^+$ has two generators $x_N(z)$ and $y_N(z)$. At least on surfaces of genus g up to $g \leq 3$ the q-expansions of the generators have integer coefficients once the leading coefficients are normalized to 1.