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Representations by some quinary quadratic forms of level 8 ☆

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ABSTRACT

Let $r_Q(n)$ be the representation number of a nonnegative integer n by a certain quinary quadratic form Q of level 8. Then the associated theta function is a modular form of weight $5/2$ for $\Gamma_0(8)$ associated with the character $\left(\frac{8}{\cdot}\right)$. We express this theta function as a linear combination of Hecke eigenforms and find the general formula of the representation number $r_Q(n)$. As a consequence, we show that $r_Q(n)$ satisfies some partially multiplicative relations by applying Fricke involution and Hecke operators on the associated theta functions.

1. Introduction

Let A be an $r \times r$ positive definite symmetric matrix over $\mathbb{Z}$ with $\det(A) = N$ such that both A and NA^{-1} have even diagonal entries. Further let

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$$Q(\mathbf{x})=\frac{1}{2}\mathbf{x}^TA\mathbf{x}\quad\text{for }\mathbf{x}=\begin{pmatrix}x_1\\ \vdots\\ x_r\end{pmatrix}\in\mathbb{Z}^r$$

be the associated quadratic form and

$$\Theta_Q(\tau)=\sum_{\mathbf{x}\in\mathbb{Z}^r}e^{2\pi iQ(\mathbf{x})\tau}=\sum_{n=0}^{\infty}r_Q(n)q^n$$

the associated theta function, where $\tau\in\mathbb{H}=\{\tau\in\mathbb{C}\mid \mathrm{Im}(\tau)>0\}$, $q=e^{2\pi i\tau}$ and

$$r_Q(n)=\#\{\mathbf{x}\in\mathbb{Z}^r\mid Q(\mathbf{x})=n\}$$

is the representation number of n by Q. When $r=2k$ is even, then $\Theta_Q(\tau)$ belongs to the space $\mathcal{M}_k(N,\chi_{(-1)^kN})$ of modular forms of weight k for $\Gamma_0(N)$ associated with the character $\chi_{(-1)^kN}=\left(\frac{(-1)^kN}{\cdot}\right)$. Suppose that $\chi_{(-1)^kN}$ is a primitive Dirichlet character modulo N. In [5] and [4], it was shown that there are only finitely many pairs (k,N) such that $\dim_{\mathbb{C}}\mathcal{M}_k(N,\chi_{(-1)^kN})=2$ and that for such pairs $\Theta_Q(\tau)$ is uniquely determined. This was proved by using the fact that for such pairs $\dim_{\mathbb{C}}\mathcal{S}_k(N,\chi_{(-1)^kN})=0$ and $\mathcal{M}_k(N,\chi_{(-1)^kN})$ are spanned by Eisenstein series with the same eigenvalues, where $\mathcal{S}_k(N,\chi_{(-1)^kN})$ is the space of cusp forms of weight k for $\Gamma_0(N)$ associated with the character $\chi_{(-1)^kN}$. Further $r_Q(n)$ satisfies

$$r_Q(p^2n)=\frac{r_Q(p^2)r_Q(n)}{r_Q(1)}$$

for any positive integer n and a prime $p\nmid Nn$ and any quadratic form of level N and rank $r=2k$.

Now let h be a positive integer and A a 5×5 positive definite symmetric integral matrix with $\det(A)=2^{2h}$ such that both A and $8A^{-1}$ have even diagonal entries. Also we denote by Q and $\Theta_Q(\tau)$ the associated quadratic form and theta function, respectively. Then $\Theta_Q(\tau)$ belongs to the space $\mathcal{M}_{\frac{5}{2}}(8,\chi_8)$ of modular forms of weight $5/2$ for $\Gamma_0(8)$ associated with the character χ_8 ([10, §2]). In this paper we shall show the following by adopting the idea in [5] and [4]. We shall first show that if $k\in\frac{1}{2}+\mathbb{Z}$ with $k\geq\frac{5}{2}$ and χ_{4N} is primitive, then $\dim_{\mathbb{C}}\mathcal{S}_k(4N,\chi_{4N})=0$ if and only if $k=\frac{5}{2}$ and $4N=8$ (Corollary 2.3(i)). From this result we shall prove that $\dim_{\mathbb{C}}\mathcal{M}_{\frac{5}{2}}(8,\chi_8)=3$ and construct the basis elements $f_1(\tau)$, $f_2(\tau)$ and $f_3(\tau)$ of $\mathcal{M}_{\frac{5}{2}}(8,\chi_8)$ by using Eisenstein series (Corollary 2.3(ii) and §4). Then we shall express $\Theta_Q(\tau)$ as a linear combination of such basis elements, namely

$$\Theta_Q(\tau)=f_1(\tau)+(2^{9-h}-8r_Q(1))f_2(\tau)+r_Q(1)f_3(\tau)$$

which implies that $\Theta_Q(\tau)$ depends only on $r_Q(1)$ (Theorem 5.2(i)). Moreover we shall provide a general formula for $r_Q(n)$ as follows. Let $n=2^{\mu}r^2t$, where μ is a nonnegative