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Identities between modular graph forms ☆

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This paper investigates the relations between modular graph forms, which are generalizations of the modular graph functions that were introduced in earlier papers motivated by the structure of the low energy expansion of genus-one Type II superstring amplitudes. These modular graph forms are multiple sums associated with decorated Feynman graphs on the world-sheet torus. The action of standard differential operators on these modular graph forms admits an algebraic representation on the decorations. First order differential operators are used to map general non-holomorphic modular graph functions to holomorphic modular forms. This map is used to provide proofs of the identities between modular graph functions for weight less than six conjectured in earlier work, by mapping these identities to relations between holomorphic modular forms which are proven by holomorphic methods. The map is further used to exhibit the structure of identities at arbitrary weight.

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1. Introduction

The structure of the low energy expansion of genus-one Type II superstring amplitudes leads one to associate modular functions with Feynman graphs for a conformal scalar field on a torus. The resulting *modular graph functions* exhibit a rich mathematical structure, which may be related to single-valued elliptic multiple polylogarithms, and single-valued multiple zeta values [9]. Although some properties of the space of modular graph functions have been investigated, a deeper understanding of their general mathematical structure is lacking.

The *weight* w of any graph is the number of scalar Green functions in the graph while the number of closed loops is denoted by L. We need only consider connected graphs since a disconnected graph is associated with a modular graph function that factors into the product of two lower weight modular graph functions. Furthermore connected graphs that are one edge reducible (that become disconnected when one edge is removed) vanish so we need to consider only connected one edge irreducible graphs.

There is a unique one-loop modular graph function with a given value of w. This is the non-holomorphic Eisenstein series E_w which satisfies the Laplace-eigenvalue equation $\Delta E_w = w(w-1)E_w$, where Δ is the Laplace–Beltrami operator on the Poincaré upper half plane. Non-holomorphic Eisenstein series are very familiar objects in number theory (for a historical account see [23]; for general overviews see for example [19,22,25]).

Two-loop modular functions were studied extensively in [10], building on earlier work in [14,13]. They were found to obey systems of inhomogeneous Laplace-eigenvalue equations whose inhomogeneous parts contain terms linear and quadratic in non-holomorphic Eisenstein series. With the help of group theoretic methods these systems were decoupled onto eigen-spaces with fixed weight and fixed eigenvalue of the Laplacian, and an infinite sequence of linear relations between modular graph functions of one- and two-loop order was shown to emerge [10].

Modular graph functions for three-loops and higher no longer systematically satisfy the type of Laplace-eigenvalue equations exhibited at one- and two-loops and, as a result, our understanding of the corresponding spaces of modular functions is much more limited. Nonetheless, a few isolated relations between the simplest modular graph functions with three and four loops were conjectured, based on the evidence of their matching Laurent expansion near the cusp [10]. The simplest of these conjectured relations, namely relating a three-loop modular graph function to one- and two-loop modular graph functions, and referred to as the D_4 conjecture, was recently proven in [11] by direct summation of the lattice sums, thereby generalizing a procedure used by Zagier at two-loops [26]. The origin of these conjectured relations and their underlying nature remained, however, unexplained.

The goal of the present paper is to make progress on various fronts by enlarging the space of non-holomorphic modular graph functions to a space of modular forms of general holomorphic and anti-holomorphic modular weights. This enlarged space will be associated with *decorated graphs* that are described as follows. Each decorated graph

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