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On perfect powers that are sums of two Fibonacci numbers ☆

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ABSTRACT

We study the equation $F_n + F_m = y^p$, where F_n and F_m are respectively the n-th and m-th Fibonacci numbers and $p \geq 2$. We find all solutions under the assumption $n \equiv m \pmod 2$.

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1. Introduction

Fibonacci numbers are prominent as well as being ancient. Their first known occurrence dates back to around 200BC, (see [5], [8]) in the earliest known treatise on *Sanskrit prosody* (poetry meters and verse in Sanskrit) entitled *Chandaḥśāstra* and authored by Piṅgala. This work is eight chapters in the late *Sūtra style* and therefore quite complex and not fully comprehensible without commentary. The Fibonacci numbers appear again (much later this time) in the work of Virahāṅka (700AD). Virahāṅka's original work has been lost, but is nevertheless cited clearly in the work of Gopāla (c. 1135); below is a translation of [9, pg. 101];

> "For four, variations of meters of two [and] three being mixed, five happens. For five, variations of two earlier – three [and] four, being mixed, eight is obtained. In this way, for six, [variations] of four [and] of five being mixed, thirteen happens. And like that, variations of two earlier meters being mixed, seven morae [is] twenty-one. In this way, the process should be followed in all mātrā–vṛttas."

The sequence is discussed rigorously and most concisely in the work of Jain scholar Acharya Hemachandra (c. 1150, living in what is known today as Gujarat) about 50 years earlier than Fibonacci's *Liber Abaci* (1202). Hemachandra, just like Piṅgala, Virahāṅka and Gopāla, was in fact studying *Sanskrit prosody* and not mathematics. Given a verse with an ending of n beats to fill, where the choice of beats consists of length 1 (called *short*) and length 2 (called *long*), in how many ways can one finish the verse? The answer lies within the fundamental sequence, defined by the recurrence;

$$H_{n+2} = H_{n+1} + H_n, \qquad H_1 = 1, \quad H_2 = 2, \quad n \geq 1, \tag{$\diamondsuit$}$$

where Hemachandra makes the concise argument that any verse that is to be filled with n beats must end with a long or a short beat. Therefore, this recurrence is enough to answer the question: given a verse with n beats remaining, one has H_n ways of finishing the *prosody*, with H_n satisfying ($\diamondsuit$).

Since the 12th century, the Hemachandra/Fibonacci numbers have sat in the spotlight of modern number theory. They have been vastly studied; intrinsically for their beautiful identities but also for their numerous applications, for example, the golden ratio has a regular appearance in art, architecture and the natural world!

Finding all perfect powers in the Fibonacci sequence was a fascinating long-standing conjecture. In 2006, this problem was completely solved by Y. Bugeaud, M. Mignotte and S. Siksek (see [4]), who innovatively combined the modular approach with classical linear forms in logarithms. In addition to this, Y. Bugeaud, F. Luca, M. Mignotte and S. Siksek also found all of the integer solutions to

$$F_n \pm 1 = y^p \quad p \geq 2, \tag{1}$$

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