Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Counting rational points on smooth cubic curves

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A R T I C L E I N F O

Article history:
Received 26 October 2016
Received in revised form 6 December 2017
Accepted 7 December 2017
Available online xxxx
Communicated by A. Pal

Keywords:
Cubic curves
Rational points
Counting function
Elliptic curves
Global determinant method
Descent

A B S T R A C T

We use a global version of Heath-Brown's p-adic determinant method developed by Salberger to give upper bounds for the number of rational points of height at most B on non-singular cubic curves defined over $\mathbb{Q}$. The bounds are uniform in the sense that they only depend on the rank of the corresponding Jacobian.

1. Introduction

Let $F(X_0, X_1, X_2) \in \mathbb{Z}[X_0, X_1, X_2]$ be a non-singular cubic form, so that $F = 0$ defines a smooth plane cubic curve C in $\mathbb{P}^2$. We want to study the asymptotic behaviour of the counting function

$$N(B) = \sharp\{P \in C(\mathbb{Q}) : H(P) \leq B\},$$

with respect to the naive height function $H(P) := \max\{|x_0|, |x_1|, |x_2|\}$ for $P = [x_0, x_1, x_2]$ with co-prime integer values of x_0, x_1, x_2.

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https://doi.org/10.1016/j.jnt.2017.12.001
0022-314X/

Please cite this article in press as: M.H. Tran, Counting rational points on smooth cubic curves, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.12.001

It is known that if the rank r of the Jacobian Jac(C) is positive, then we have

$$N(B) \sim c_F (\log B)^{r/2} \tag{1}$$

as $B \to \infty$. This result was shown by Néron. Moreover, if $r = 0$ then $N(B) \leq 16$ by Mazur's theorem (see Mazur [5], Theorem 8) on torsion groups of elliptic curves. But (1) is not a uniform upper bound as the constant c_F depends on C. The aim of this paper is to give uniform upper bounds for $N(B)$ which only depend on the rank of Jac(C).

In this direction, Heath-Brown and Testa (see [4], Corollary 1.3) established the uniform bound

$$N(B) \ll (\log B)^{3+r/2} \tag{2}$$

by using the p-adic determinant method developed by the first author (see [3]). In [4], they also used a result of David [1] about the successive minima of the quadratic form given by the canonical height pairing on Jac(C) to prove the sharper uniform bounds $N(B) \ll (\log B)^{1+r/2}$ for all r and $N(B) \ll (\log B)^{r/2}$ if r is sufficiently large.

We shall in this paper give a direct proof of the bound

$$N(B) \ll (\log B)^{2+r/2}, \tag{3}$$

based on the determinant method, which does not depend on any deep result about the canonical height pairing.

To do this, we follow the approach in [4] with descent. But we replace the p-adic determinant method by a global determinant method developed by Salberger [6]. The main result of this paper is the following

Theorem 1. *Let $F(X_0, X_1, X_2) \in \mathbb{Z}[X_0, X_1, X_2]$ be a non-singular cubic form, so that $F = 0$ defines a smooth plane cubic curve C. Let r be the rank of Jac(C). Then for any $B \geq 3$ and any positive integer m we have*

$$N(B) \ll m^r \left(B^{\frac{2}{3m^2}} + m^2\right) \log B$$

uniformly in C, with an implied constant independent of m.

This bound improves upon the estimate

$$N(B) \ll m^{r+2} \left(B^{\frac{2}{3m^2}} \log B + \log^2 B\right)$$

in [4] (see Theorem 1.2). Taking $m = 1 + [\sqrt{\log B}]$ we immediately obtain the following result.

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