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General order Euler sums with multiple argument

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We provide an explicit analytical representation for Euler type sums of harmonic numbers with multiple arguments. We also explore the representation of integrals with logarithmic and hypergeometric integrand in terms of the polygamma function and other special functions. The integrals in question will be associated with harmonic numbers of positive terms. A few examples of integrals will be given an identity in terms of some special functions including the Riemann zeta function.

1. Introduction

The study of the representation of infinite series in closed form goes back to antiquity, and it was the genius Euler who put the investigation of the analysis of series on a firm foundation. The celebrated Basel problem introduced the zeta function and Euler went on to represent binomial and harmonic number series of the type $\sum_{n\geq 1} \frac{H_n^{(m)}}{n^q}$ in closed form. The famous Euler recurrence expression states

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$$2\sum_{n\geq 1}\frac{H_n}{n^m}=(m+2)\,\zeta(m+1)-\sum_{j=1}^{m-2}\zeta(j+1)\,\zeta(m-j)\,,$$

see [4]. Euler's harmonic number series have been extended and variations have been investigated by [1], [5], [10], [15], [18], [24] and many others. Binomial, inverse binomial and harmonic number series are of interest to physics and have been studied in order to perform calculations of higher order corrections to scattering processes in particle physics, see [9], [22], [28] and [45]. In [2], the authors explore the algorithmic and analytic properties of so-called generalized harmonic sums systematically, in order to compute the massive Feynman integrals which arise in quantum field theories and in certain combinatorial problems. Hence there is great motivation to study the representation of Euler series in closed form: a good account on closed form, what they are and why we care has been eloquently enunciated in [11]. There exists, in the literature, see [14], [23], [32] results on sums of the form $\sum_{n\geq 1}\frac{H_n^{(m)}}{n\binom{n+k}{k}}$, but very many fewer results incorporating series of harmonic numbers with multiple argument $H_{pn}^{(m)}$. A search of the literature yields no closed form representation of $T(k,m,p)=\sum_{n\geq 1}\frac{H_{pn}^{(m)}}{n\binom{n+k}{k}}$ for $p\geq 3$. In this paper we will develop analytical representations for Euler type series with inverse binomial coefficients and harmonic numbers of the type $T(k,m,p)$. The extra parameter p serves to unify a large number of previously published results, see [34], [41], [49], [46] and references therein. Also, by association, we are able to demonstrate a representation for integrals of the type $\int_0^1 \frac{(1-x^p)}{x^p(1-x)}\ln^{m-1}x\ln(1-x^p)\,dx$.

2. Preliminaries

We define a harmonic number with multiple argument as H_{pn} for $p\in\mathbb{N}\backslash\{1\}$; $\mathbb{N}:=\{1,2,3,\cdots\}$, the set of natural numbers. For $p=1$, we write H_n as the nth harmonic number with unitary argument. In this paper we will develop analytical representations for Euler type sums with inverse binomial coefficients of the type

$$T(k,m,p)=\sum_{n\geq 1}\frac{H_{pn}^{(m)}}{n\binom{n+k}{k}} \tag{2.1}$$

for $(m,k,p)\in\mathbb{N}$. Furthermore we discuss analytical representations of the integral

$$\int_0^1 \frac{x^p\ln^{m-1}x}{1-x}\,{}_2F_1\left[\left.\begin{matrix}1,1\\2+k\end{matrix}\right|x^p\right]dx \tag{2.2}$$

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