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On a sign pattern for the crank function of cubic partition pairs

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ABSTRACT

In a recent paper, B. Kim and P.-C. Toh conjectured a sign pattern of a certain arithmetic function on the crank function of cubic partition pairs. We confirm their conjecture by obtaining its asymptotic formula.

1. Introduction and statement of results

In [11], H. Zhao and Z. Zhong studied the number of cubic partition pairs $b(n)$ defined by

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2},$$

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where $(a;q)_\infty$ is the standard q-product defined by $(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1})$. They investigated congruence properties modulo 5, 7, and 9. For example,

$$b(5n+4) \equiv 0 \pmod 5. \tag{1.1}$$

To prove (1.1) combinatorially, Kim and Toh [9] introduced the crank function $M(m,n)$ of cubic partition pair $b(n)$:

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)z^m q^n := \left(\frac{(q;q)_\infty (q^2;q^2)_\infty}{(zq;q)_\infty (q/z;q)_\infty (zq^2;q^2)_\infty (q^2/z;q^2)_\infty}\right)^2. \tag{1.2}$$

They proved that, for all nonnegative integer n,

$$M(0,5,5n+4) \equiv M(1,5,5n+4) \equiv \cdots \equiv M(4,5,5n+4) \equiv 0 \pmod 5$$

where $M(k,j,n) := \sum_{m\equiv k \pmod j} M(m,n)$ is the number of cubic partition pairs of n with crank $\equiv k \pmod j$. Moreover, the following arithmetic properties for the crank function $M(m,n)$ were studied:

$$\begin{aligned} M(1,4,n) &= M(3,4,n), \\ M(0,4,2n+1) &\equiv M(1,4,2n+1) \pmod 4, \\ M(1,4,n) &\equiv M(2,4,n) \pmod 2. \end{aligned} \tag{1.3}$$

In the same article [9], they conjectured a sign pattern of $M(0,4,n) - M(2,4,n)$ with period 16. See Corollary 1.2 for their conjecture. For the ordinary partition function, there have been extensive studies on the sign patterns for the rank and crank functions. G. E. Andrews and F. G. Lewis [2] investigated the difference between the rank functions and the crank functions modulo 2 and 4 and conjectured sign patterns for ordinary partition rank function modulo 3 and ordinary partition crank function modulo 3 and 4. The conjectures on the ordinary partition crank function were proved by D. M. Kane [8] and O-Y. Chan [6]. The sign patterns for the ordinary partition rank function were proved by K. Bringmann [5] asymptotically. Later, S. H. Chan and R. Mao [7] proved the conjecture on the rank functions using q-series techniques which do not require carrying out any extensive computations.

Our main aim is confirming Kim and Toh's conjecture. We prove the following asymptotic formula of $M(0,4,n) - M(2,4,n)$ which confirms the conjecture asymptotically. By checking the first few coefficients using a computer, we confirm their conjecture in Corollary 1.2.

Theorem 1.1. *For $n > 80$,*

$$\begin{aligned} &M(0,4,n) - M(2,4,n) \\ &= \left(2\pi \cos\frac{(n+1)\pi}{2} \cos\frac{(2n-7)\pi}{8}\right)\sqrt{n-\frac{1}{4}}\, I_1\left(\frac{\pi}{4}\sqrt{n-\frac{1}{4}}\right) \end{aligned}$$