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# Full Dimensional Sets of Reals Whose Sums of Partial Quotients Increase in Certain Speed

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ABSTRACT

For a real $x \in (0,1) \setminus \mathbb{Q}$, let $x = [a_1(x), a_2(x), \cdots]$ be its continued fraction expansion. Let $s_n(x) = \sum_{j=1}^{n} a_j(x)$. The Hausdorff dimensions of the level sets

$$E_{\varphi(n),\alpha} := \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{s_n(x)}{\varphi(n)} = \alpha \right\}$$

for $\alpha \geq 0$ and a non-decreasing sequence $\{\varphi(n)\}_{n=1}^{\infty}$ have been studied by E. Cesaratto, B. Vallée, J. Wu, J. Xu, G. Iommi, T. Jordan, L. Liao, M. Rams et al. In this work we carry out a kind of inverse project of their work, that is, we consider the conditions on $\varphi(n)$ under which one can expect a 1-dimensional set $E_{\varphi(n),\alpha}$. We give certain upper and lower bounds on the increasing speed of $\varphi(n)$ when $E_{\varphi(n),\alpha}$ is of Hausdorff dimension 1 and a new class of sequences $\{\varphi(n)\}_{n=1}^{\infty}$ such that $E_{\varphi(n),\alpha}$ is of full dimension. For an irregular sequence $\{\varphi(n)\}_{n=1}^{\infty}$, a full dimensional set $E_{\varphi(n),\alpha}$ is impossible.



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## 1. Introduction

Let $I = [0, 1]$ be the unit interval. For a real $x \in I \setminus \mathbb{Q}$, let

$$x = [a_1(x), a_2(x), a_3(x), \cdots] := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} \tag{1.1}$$

be its continued fraction expansion. Let

$$G(x) = \left\{ \frac{1}{x} \right\} = \frac{1}{x} - \left[ \frac{1}{x} \right]$$

be the Gauss map, with the two symbols { } and [ ] being the fractional and integral part of the number. $G(x)$ is conjugated to a shift map on a countable alphabet. Let

$$s_n(x) = \sum_{j=1}^{n} a_j(x)$$

be the sum of the first $n$ partial quotients, $n \in \mathbb{N}$. We focus on the limit behaviors of $s_n(x)$ in this work. According to A.Ya. Khinchin [Khi],

$$\lim_{n\to\infty} \frac{s_n(x)}{n} = \infty$$

almost everywhere with respect to Lebesgue measure. In 1988, W. Philipp [Phi, Theorem 1] strengthened Khinchin's result by showing that, for a sequence of positive numbers $\varphi(n)$ such that $\frac{\varphi(n)}{n}$ is non-decreasing,

$$\lim_{n\to\infty} \frac{s_n}{\varphi(n)} = 0 \text{ or } \limsup_{n\to\infty} \frac{s_n}{\varphi(n)} = \infty \text{ a. e.}$$

according to whether $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$ or $= \infty$. His proof relies on the theory of mixing random vectors or triangular arrays. As to subsets of the residual set, which are all of measure 0, it turns out that the Hausdorff dimension is a useful tool to distinguish their sizes. The initial explorer of Hausdorff dimensions of related sets is A. Besicovitch [Bes]. Since then there are continuously other dimensional results, from the distribution of various terms to regularity of these distributions with respect to various levels. It turns out that the dimensional results in continued fractions provide plentiful and intricate examples and tests for various theories of fractal geometry [Man].

From the point view of dynamical systems $(I, G(x))$, E. Cesaratto and B. Vallée [CV], G. Iommi and T. Jordan [IJ1,IJ2] got interesting results on Hausdorff dimensions of the sets

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