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Refined dimensions of cusp forms, and equidistribution and bias of signs

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We refine known dimension formulas for spaces of cusp forms of squarefree level, determining the dimension of subspaces generated by newforms both with prescribed global root numbers and with prescribed local signs of Atkin–Lehner operators. This yields precise results on the distribution of signs of global functional equations and sign patterns of Atkin–Lehner eigenvalues, refining and generalizing earlier results of Iwaniec, Luo and Sarnak. In particular, we exhibit a strict bias towards root number +1 and a phenomenon that sign patterns are biased in the weight but perfectly equidistributed in the level. Another consequence is lower bounds on the number of Galois orbits.

0. Introduction

Let $S_k^{\mathrm{new}}(N)$ denote the new subspace of weight k elliptic cusp forms on $\Gamma_0(N)$. Dimensions of such spaces are well known (cf. [Mar05]). If $N > 1$, one can decompose, in various ways, $S_k^{\mathrm{new}}(N)$ into certain natural subspaces. A crude decomposition is into the plus and minus spaces $S_k^{\mathrm{new},\pm}(N)$, which are the subspaces of $S_k^{\mathrm{new}}(N)$ generated

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by newforms with global root number ± 1, i.e., sign $\pm$ in the functional equation of their L-functions. A more refined decomposition is to consider subspaces generated by newforms with fixed local components π_p at primes $p|N$, where each π_p is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ of conductor $p^{v_p(N)}$.

In this paper, we obtain explicit dimension formulas for both of these types of subspaces in the case $N > 1$ is squarefree. This case is simple because there are only two possibilities for the local components π_p, the Steinberg representation and its unramified quadratic twist, and distinguishing between these two is equivalent to specifying the Atkin–Lehner eigenvalue. The proof relies on a trace formula for products of Atkin–Lehner operators on $S_k(N)$ due to Yamauchi [Yam73], which we translate to $S_k^{\mathrm{new}}(N)$ in Section 1. It was already known that such dimensions can be computed in principle in this way, and some cases have been done before: the prime level case is in [Wak14], asymptotics for dimensions of plus and minus spaces were given in [ILS00], and [HH95] gave a formula for the full cusp space in level 2 with prescribed Atkin–Lehner eigenvalues. So while the derivation is not especially novel, we hope the explicit formulas and their consequences (particularly the biases discussed below) may be of interest. In fact, our motivation was different from [HH95], [ILS00] and [Wak14], which all had mutually distinct motivations.

We emphasize that we are able to obtain quite simple formulas thanks to the squarefree assumption. The trace formula in [Yam73] is valid for arbitrary level, but becomes considerably more complicated. In principle, our approach gives dimensions of spaces with prescribed Atkin–Lehner eigenvalues in non-squarefree level also, but the resulting formulas may be messy. In any case, this would not give us dimensions for forms with specified local components π_p for non-squarefree levels.

Our motivation in computing these dimensions comes from two sources. First, this allows us to get very precise results about the distribution of signs of global functional equations and sign patterns for collections of Atkin–Lehner operators. Various equidistribution results are known about local components at unramified places, or equivalently, Hecke eigenvalues at primes away from the level. For instance, and perhaps most analogous, distributions of signs of unramified Hecke eigenvalues are considered in [KLSW10]. At ramified places, the most general results we know of for GL(2) are by Weinstein [Wei09], where he proves equidistribution of local inertia types for general level as the weight tends to infinity. However, these local inertia types do not distinguish unramified quadratic twists, and thus give no information in the case of squarefree level. While the fact that equidistribution holds is not at all surprising, the precise results we obtain about distribution and bias were perhaps not expected. Let us explain this in more detail.

For the rest of the paper, assume $N > 1$ is squarefree and $k \geq 2$ is even.

In Section 2 we obtain the dimension formulas for the plus and minus spaces. This implies the root number is equidistributed between $+1$ and -1 and the difference between the dimensions of the plus and minus spaces is essentially independent of k. It is also subpolynomial in N—precisely $O(2^{\omega(N)})$, where $\omega(N)$ is the number of prime