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Estimates of sums related to the Nyman–Beurling criterion for the Riemann Hypothesis

Helmut Maier [a], Michael Th. Rassias [b,c,d,*]

[a] *Department of Mathematics, University of Ulm, Helmholtzstrasse 18, 89081 Ulm, Germany*
[b] *Institute of Mathematics, University of Zurich, CH-8057, Switzerland*
[c] *Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Institutskiy per, d. 9, Russia*
[d] *Institute for Advanced Study, Program in Interdisciplinary Studies, 1 Einstein Dr, Princeton, NJ 08540, USA*

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We give an estimate for sums appearing in the Nyman–Beurling criterion for the Riemann Hypothesis containing the Möbius function. The estimate is remarkably sharp in comparison to estimates of other sums containing the Möbius function. The methods intensively use tools from the theory of continued fractions and from the theory of Fourier series.

* Corresponding author.
E-mail addresses: helmut.maier@uni-ulm.de (H. Maier), michail.rassias@math.uzh.ch, rassias@ias.edu (M.Th. Rassias).

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1. Introduction

According to the approach of Nyman–Beurling–Báez–Duarte (see [1], [6], [8]) to the Riemann Hypothesis, the Riemann Hypothesis is true if and only if

$$\lim_{N\to\infty} d_N^2 = 0\,,$$

where

$$d_N^2 = \inf_{D_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta\left(\frac{1}{2}+it\right) D_N\left(\frac{1}{2}+it\right)\right|^2 \frac{dt}{\frac{1}{4}+t^2} \tag{1.1}$$

and the infimum is over all Dirichlet polynomials

$$D_N(s) := \sum_{n=1}^{N} \frac{a_n}{n^s}\,,\ a_n \in \mathbb{C}\,,$$

of length N (see [7]).

It follows from [7] that under certain assumptions among all Dirichlet polynomials $D_N(s)$ the infimum in (1.1) is attained for $D_N(s) = V_N(s)$, where

$$V_N(s) := \sum_{n=1}^{N} \left(1 - \frac{\log n}{\log N}\right) \frac{\mu(n)}{n^s}\,. \tag{1.2}$$

It thus is of interest to obtain an unconditional estimate for the integral in (1.1).

If we expand the square in (1.1) we obtain

$$\begin{aligned} d_N^2 = \inf_{D_N} \Bigg(\int_{-\infty}^{\infty} \left(1 - \zeta\left(\frac{1}{2}+it\right) D_N\left(\frac{1}{2}+it\right) - \zeta\left(\frac{1}{2}-it\right) \overline{D}_N\left(\frac{1}{2}+it\right)\right) \frac{dt}{\frac{1}{4}+t^2} \\ + \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|D_N\left(\frac{1}{2}+it\right)\right|^2 \frac{dt}{\frac{1}{4}+t^2} \Bigg)\,. \end{aligned}$$

The last integral evaluates as

$$\sum_{1\le h,k\le N} a_h \bar{a}_k h^{-1/2} k^{-1/2} \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left(\frac{h}{k}\right)^{it} \frac{dt}{\frac{1}{4}+t^2}\,.$$

We have