ARTICLE IN PRESS YJNTH:5994

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Universal sums of generalized octagonal numbers

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A R T I C L E I N F O

Article history:
Received 28 April 2017
Received in revised form 28 September 2017
Accepted 8 December 2017
Available online xxxx
Communicated by M. Pohst

MSC:
11E12
11E20

Keywords:
Lagrange's four square theorem
Generalized octagonal numbers

A B S T R A C T

An integer of the form $P_8(x) = 3x^2 - 2x$ for some integer x is called a generalized octagonal number. A quaternary sum $\Phi_{a,b,c,d}(x, y, z, t) = aP_8(x) + bP_8(y) + cP_8(z) + dP_8(t)$ of generalized octagonal numbers is called *universal* if $\Phi_{a,b,c,d}(x, y, z, t) = n$ has an integer solution x, y, z, t for any positive integer n. In this article, we show that if $a = 1$ and $(b, c, d) = (1, 3, 3), (1, 3, 6), (2, 3, 6), (2, 3, 7)$ or $(2, 3, 9)$, then $\Phi_{a,b,c,d}(x, y, z, t)$ is universal. These were conjectured by Sun in [10]. We also give an effective criterion on the universality of an arbitrary sum $a_1P_8(x_1) + a_2P_8(x_2) + \cdots + a_kP_8(x_k)$ of generalized octagonal numbers, which is a generalization of "15-theorem" of Conway and Schneeberger.

1. Introduction

The famous Lagrange's four square theorem states that every positive integer can be written as a sum of at most four integral squares. Motivated by Lagrange's four

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[1] The work of the first author was supported by BK21 PLUS SNU Mathematical Sciences Division.
[2] The work of the second author was supported by the National Research Foundation of Korea (NRF-2017R1A2B4003758).

https://doi.org/10.1016/j.jnt.2017.12.014
0022-314X/

Please cite this article in press as: J. Ju, B.-K. Oh, Universal sums of generalized octagonal numbers, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.12.014

square theorem, Ramanujan provided a list of 55 candidates of diagonal quaternary quadratic forms that represent all positive integers. In [4], Dickson pointed out that the quaternary form $x^2 + 2y^2 + 5z^2 + 5t^2$ that is included in Ramanujan's list represents all positive integers except 15 and confirmed that Ramanujan's assertion for all the other 54 forms is true. A positive-definite integral quadratic form is called *universal* if it represents all non-negative integers. The problem on determining all universal quaternary forms was completed by Conway and Schneeberger. They proved that there are exactly 204 universal quaternary quadratic forms. Furthermore, they proved the so called "15-theorem", which states that every positive-definite integral quadratic form that represents $1, 2, 3, 5, 6, 7, 10, 14$ and 15 is, in fact, universal, irrespective of its rank (see [1]). Recently, Bhargava and Hanke [2] proved the so-called, "290-theorem", which states that every positive-definite integer-valued quadratic form is universal if it represents

$$1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 13,\ 14,\ 15,\ 17,\ 19,\ 21,\ 22,\ 23,\ 26,\ 29,$$
$$30,\ 31,\ 34,\ 35,\ 37,\ 42,\ 58,\ 93,\ 110,\ 145,\ 203,\ \text{ and }\ 290.$$

Here a quadratic form $f(x_1, x_2, \ldots, x_n) = \sum_{1\leqslant i,j\leqslant n} a_{ij}x_ix_j$ $(a_{ij} = a_{ji})$ is called *integral* if $a_{ij} \in \mathbb{Z}$ for any i, j, and is called *integer-valued* if $a_{ii} \in \mathbb{Z}$ and $a_{ij} + a_{ji} \in \mathbb{Z}$ for any i, j.

In general, a *polygonal number of order m* (or *an m-gonal number*) for $m \geqslant 3$ is defined by

$$P_m(x) = \frac{(m-2)x^2 - (m-4)x}{2}$$

for some non-negative integer x. If we admit that x is a negative integer, then $P_m(x)$ is called *a generalized polygonal number of order m* (or *a generalized m-gonal number*). Then, Lagrange's four square theorem implies that the diophantine equation

$$P_4(x) + P_4(y) + P_4(z) + P_4(t) = n$$

has an integer solution x, y, z and t, for any non-negative integer n.

Recently, Sun in [10] proved that every positive integer can be written as a sum of four generalized octagonal numbers, which is also considered as a generalization of Lagrange's four square theorem. He also defined that for positive integers $a \leqslant b \leqslant c \leqslant d$, a quaternary sum $aP_8(x) + bP_8(y) + cP_8(z) + dP_8(t)$ (simply, $\Phi_{a,b,c,d}(x, y, z, t)$) of generalized octagonal numbers is *universal over $\mathbb{Z}$* if the diophantine equation

$$\Phi_{a,b,c,d}(x, y, z, t) = aP_8(x) + bP_8(y) + cP_8(z) + dP_8(t) = n$$

has an integer solution for any non-negative integer n. Then, he showed that if $\Phi_{a,b,c,d}$ is universal over $\mathbb{Z}$, then $a = 1$, and (b, c, d) is one of the following 40 triples: