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Products of Eisenstein series and Fourier expansions of modular forms at cusps

Martin Dickson, Michael Neururer [a,*]

[a] *Technische Universität Darmstadt, Schlossgartenstraße 7, 64289 Darmstadt, Germany*

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A B S T R A C T

We show, for levels of the form $N = p^a q^b N'$ with N' squarefree, that in weights $k \geq 4$ every cusp form $f \in \mathcal{S}_k(N)$ is a linear combination of products of certain Eisenstein series of lower weight. In weight $k = 2$ we show that the forms f which can be obtained in this way are precisely those in the subspace generated by eigenforms g with $L(g, 1) \neq 0$. As an application of such representations of modular forms we can calculate Fourier expansions of modular forms at arbitrary cusps and we give several examples of such expansions in the last section.

1. Introduction

The space $\mathcal{M}_k(N, \chi)$ of modular forms of level $\Gamma_0(N)$, weight k and nebentypus χ splits into the direct sum of the Eisenstein subspace $\mathcal{E}_k(N, \chi)$ and the space of cusp forms $\mathcal{S}_k(N, \chi)$. It is straightforward to compute Fourier expansions and Hecke eigenforms in the Eisenstein subspace, but the space of cusp forms is far more mysterious, and any method of generating cusp forms is therefore of great interest. In this article we examine

* Corresponding author.
E-mail address: neururer@mathematik.tu-darmstadt.de (M. Neururer).

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one of the simplest methods of generating cusp forms: What is the subspace of $\mathcal{S}_k(N,\chi)$ generated by (the cuspidal projection of) products of Eisenstein series?

For $N = 1$ the answer to this question is very well-known: the graded ring[1] $\oplus_{k\geq 0}\mathcal{M}_k(1)$ is a polynomial ring with two generators, one in degree four and one in degree six, corresponding to the Eisenstein series E_4 and E_6. This means that every cusp form of level $N = 1$ is a linear combination of products of Eisenstein series. However the number of products required to form the monomials in E_4 and E_6 for these linear combinations grows linearly with the weight k, which means these monomials are rather complicated. It is therefore natural to ask whether one can have simpler products at the expense of taking more Eisenstein series. Pushing this to the extreme we are led to ask: What is the subspace of $\mathcal{M}_k(N,\chi)$ generated by Eisenstein series and products of *two* Eisenstein series?

Using the Rankin–Selberg method (as observed in [23] §5) one can show that, for $k \geq 8$, $\mathcal{M}_k(1)$ can be generated by the products $E_l E_{k-l}$ for $4 \leq l \leq k-4$. Similar statements are known to hold for $\mathcal{M}_k(p)$ for p prime and $k \geq 4$ (see Imamoḡlu–Kohnen [10] for $p = 2$, Kohnen–Martin [12] for $p > 2$). The most complete result in this direction was found by Borisov–Gunnells in [3], [2], and [4]. They show that for weights greater than two and any level $N \geq 1$ the whole space $\mathcal{M}_k(\Gamma_1(N))$ is generated by $\mathcal{E}_k(\Gamma_1(N))$ and products of two *toric* Eisenstein series $\tilde{s}^{(k)}_{a/N}$ for $a \in \{0,\dots,N-1\}$, while for $k = 2$ one only obtains a subspace, $\mathcal{S}_{2,\mathrm{rk}=0}(N) + \mathcal{E}_k(N,\chi)$, where $\mathcal{S}_{2,\mathrm{rk}=0}(N)$ is defined below.

The main application we want to present is the calculation of Fourier expansions at arbitrary cusps. While the toric Eisenstein series of Borisov–Gunnells have remarkably simple rational Fourier expansions at ∞, the Fourier expansions at other cusps in weights higher than 1 are harder to obtain. This led us to consider instead the well-studied Eisenstein series

$$E_l^{\phi,\psi}(z) = e_l^{\phi,\psi} + 2\sum_{n\geq 1}\sigma_{l-1,\phi,\psi}(n)q^n \in \mathcal{M}_l(M,\phi\psi), \tag{1}$$

where $q = e^{2\pi i z}$, ϕ and ψ are primitive characters of level M_1 and M_2, $M_1M_2 = M \mid N$, $\sigma_{l-1,\phi,\psi}(n) = \sum_{d|n}\phi(n/d)\psi(d)d^{l-1}$, and the constant term $e_l^{\phi,\psi}$ (either zero or a value of a Dirichlet L-function) is given in Section 4. The advantage of working directly with the Eisenstein series in (1) is that their Fourier expansions at cusps other than ∞ are comparatively easy to obtain and were explicitly calculated by Weisinger [21] (we use a corrected version by Cohen [6]).

Before we describe our results we mention a different, rather general recent result by Raum [22]: Let $k \geq 8$ be an integer, let ρ be a representation of $\mathrm{SL}_2(\mathbb{Z})$ on a complex vector space V such that $\ker(\rho)$ contains a congruence subgroup, and define $\mathcal{M}_k(\rho)$ to be the space of V-valued functions transforming as modular forms for the automorphy factor $\gamma \mapsto (cz+d)^{-k}\rho(\gamma^{-1})$. Then

[1] When $\chi = \mathbf{1}_N$ is the principal character modulo N we write $\mathcal{M}_k(N)$ for $\mathcal{M}_k(N,\mathbf{1}_N)$.

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