ARTICLE IN PRESS YJNTH:5955

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Bilinear forms with exponential sums with binomials

Kui Liu[a], Igor E. Shparlinski[b], Tianping Zhang[c,*]

[a] School of Mathematics and Statistics, Qingdao University, No. 308, Ningxia Road, Shinan, Qingdao, Shandong, 266071, PR China
[b] Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia
[c] School of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710019 Shaanxi, PR China

A R T I C L E I N F O

Article history:
Received 29 October 2017
Received in revised form 12 December 2017
Accepted 19 December 2017
Available online xxxx
Communicated by S.J. Miller

MSC:
11D79
11L07

Keywords:
Binomial sums
Cancellation
Bilinear form

A B S T R A C T

We obtain several estimates for bilinear forms with exponential sums with binomials $mx^k + nx^\ell$. In particular we show the existence of nontrivial cancellations between such sums when the coefficients m and n vary over rather sparse sets of general nature.

* Corresponding author.
E-mail addresses: liukui@qdu.edu.cn (K. Liu), igor.shparlinski@unsw.edu.au (I.E. Shparlinski), tpzhang@snnu.edu.cn (T.P. Zhang).

https://doi.org/10.1016/j.jnt.2017.12.011
0022-314X/© 2018 Published by Elsevier Inc.

Please cite this article in press as: K. Liu et al., Bilinear forms with exponential sums with binomials, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.12.011

1. Introduction

1.1. Background and motivation

For a positive integer q, we denote by $\mathbb{Z}_q$ the residue ring modulo q and also denote by $\mathbb{Z}_q^*$ the group of units of $\mathbb{Z}_q$.

For fixed integers k and ℓ, we consider exponential sums with binomials

$$S_{k,\ell,q}(m,n) = \sum_{x \in \mathbb{Z}_q^*} \mathbf{e}_q\left(mx^k + nx^\ell\right),$$

where for negative powers of x are computed modulo q and

$$\mathbf{e}_q(z) = \exp(2\pi i z/q).$$

The case $(k,\ell) = (1,-1)$ corresponds to the case of Kloosterman sums. We note that when both k and ℓ are positive there is no reason to restrict the summation to $\mathbb{Z}_q^*$. However, motivated by the choice $(k,\ell) = (-2,1)$ which is important for applications to square-free numbers in progressions, see [13] we only consider this case. It is also important for the validity of the bound (1.3) below.

Furthermore, given two sets $\mathcal{M}, \mathcal{N} \subseteq \mathbb{Z}_q$ and two sequences of weights $\boldsymbol{\alpha} = \{\alpha_m\}_{m\in\mathcal{M}}$ and $\boldsymbol{\beta} = \{\beta_n\}_{n\in\mathcal{N}}$, we define the following bilinear forms with the binomial sums $S_{k,\ell,q}(m,n)$:

$$\mathcal{S}_{k,\ell,q}(\boldsymbol{\alpha},\boldsymbol{\beta};\mathcal{M},\mathcal{N}) = \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} \alpha_m\beta_n S_{k,\ell,q}(m,n).$$

We also consider the following special cases

$$\begin{aligned}\mathcal{S}_{k,\ell,q}(\boldsymbol{\alpha};\mathcal{M},\mathcal{N}) &= \mathcal{S}_{k,\ell,q}\left(\boldsymbol{\alpha},\{1\}_{n\in\mathcal{N}};\mathcal{M},\mathcal{N}\right) \\ &= \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} \alpha_m S_{k,\ell,q}(m,n),\end{aligned} \tag{1.1}$$

and

$$\begin{aligned}\mathcal{S}_{k,\ell,q}(\mathcal{M},\mathcal{N}) &= \mathcal{S}_{k,\ell,q}\left(\{1\}_{m\in\mathcal{M}},\{1\}_{n\in\mathcal{N}};\mathcal{M},\mathcal{N}\right) \\ &= \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} S_{k,\ell,q}(m,n).\end{aligned} \tag{1.2}$$

For $(k,\ell) = (1,-1)$, that is, for Kloosterman sums, such bilinear forms have been introduced by Fouvry, Kowalski and Michel [3] who have also demonstrated the importance of estimating them beyond of what follows immediately from the Weil bound of Kloosterman sums (see, for example, [7, Chapter 11]), that is, better than the bound (1.3) below.

Please cite this article in press as: K. Liu et al., Bilinear forms with exponential sums with binomials, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.12.011