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Distributions of full and non-full words in beta-expansions

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The structures of full words and non-full for β-expansions are completely characterized in this paper. We obtain the precise lengths of all the maximal runs of full and non-full words among admissible words with same order.

1. Introduction

Let $\beta > 1$ be a real number. The β-expansion was introduced by Rényi [Ren57] in 1957, which generalized the usual decimal expansions (generally N-adic expansion with integers $N > 1$) to that with any real base β. There are some different behaviors for

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the representations of real numbers and corresponding dynamics for the integer and noninteger cases. For example, when $\beta \in \mathbb{N}$, every element in $\{0, 1, \cdots, \beta - 1\}^{\mathbb{N}}$ (except countablely many ones) is the β-expansion of some $x \in [0, 1)$ (called admissible sequence). However, if $\beta \notin \mathbb{N}$, not any sequence in $\{0, 1, \cdots, \lfloor\beta\rfloor\}^{\mathbb{N}}$ is the β-expansion of some $x \in [0, 1)$ where $\lfloor\beta\rfloor$ denotes the integer part of β. Parry [Pa60] managed to provide a criterion for admissability of sequences (see Lemma 2.3 below). Any finite truncation of an admissible sequence is called an admissible word. Denoted by Σ_β^n the set of all admissible words with length $n \in \mathbb{N}$. By estimating the cardinality of Σ_β^n in [Ren57], it is known that the topological entropy of β-transformation T_β is $\log\beta$. The projection of any word in Σ_β^n is a cylinder of order n (also say a fundamental interval), which is a left-closed and right-open interval in $[0, 1)$. The lengths of cylinders are irregular for $\beta \notin \mathbb{N}$, meanwhile, they are all regular for $\beta \in \mathbb{N}$, namely, the length of any cylinder of order n equals β^{-n}. Li and Wu [LiWu08] introduced a classification of $\beta > 1$ for characterizing the regularity of the lengths of cylinders and then the sizes of all corresponding classes were given by Li, Persson, Wang and Wu [LPWW14] in the sense of measure and dimension. Another different classification of $\beta > 1$ was provided by Blanchard [Bla89] from the viewpoint of dynamical system, and then the sizes of all corresponding classes were given by Schmeling [Schme97] in the sense of topology, measure and dimension (see [TaWa11], [TWWX13], [BaLi14] for more research on beta-expansions from the viewpoint of dynamical system).

A cylinder with order n is said to be full if it is mapped by the n-th iteration of β-transformation T_β^n onto $[0, 1)$ (see Definition 2.6 below, [Wal78] or [DK02]) or equivalently its length is maximal, that is, equal to β^{-n} (see Proposition 3.1 below, [FW12] or [BuWa14]). An admissible word is said to be full if the corresponding cylinder is full. Full words and cylinders have very good properties. For example, Walters [Wal78] proved that for any given $N > 0$, $[0, 1)$ is covered by the full cylinders of order at least N. Fan and Wang [FW12] obtained some good properties of full cylinders (see Proposition 3.1 and Proposition 3.2 below). Bugeaud and Wang [BuWa14] studied the distribution of full cylinders, showed that for $n \geq 1$, among every $(n+1)$ consecutive cylinders of order n, there exists at least one full cylinder, and used it to prove a modified mass distribution principle to estimate the Hausdorff dimension of sets defined in terms of β-expansions. Zheng, Wu and Li proved that the extremely irregular set is residual with the help of the full cylinders (for details see [ZWL17]).

In this paper, we are interested in the distributions of full and non-full words in Σ_β^n, i.e., the distributions of full and non-full cylinders in $[0, 1)$. More precisely, we consider the lexicographically ordered sequence of all order n admissible words, and count the numbers of successive full words and successive non-full words. Or, in what amounts to the same thing, we look at all the fundamental intervals of order n, arranged in increasing order along the unit interval, and ask about numbers of successive intervals where T_β^n is onto (and numbers of intervals where it is not onto). Our main results concern the maximal number of successive full words, and the maximal number of successive non-full