Journal of Number Theory 190 (2018) 333–351

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Mordell–Weil lattice of Inose's elliptic $K3$ surface arising from the product of 3-isogenous elliptic curves

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A R T I C L E   I N F O

Article history:
Received 19 October 2016
Received in revised form 6 February 2018
Accepted 13 March 2018
Available online 24 April 2018
Communicated by S.J. Miller

MSC:
primary 14J27, 14J28
secondary 14H52, 11G05

Keywords:
$K3$ surface
Elliptic surface
Elliptic curve

A B S T R A C T

From the product of two elliptic curves, Shioda and Inose [6] constructed an elliptic $K3$ surface having two II^* fibers. Its Mordell–Weil lattice structure depends on the morphisms between the two elliptic curves. In this paper, we give a method of writing down generators of the Mordell–Weil lattice of such elliptic surfaces when two elliptic curves are 3-isogenous. In particular, we obtain a basis of the Mordell–Weil lattice for the singular $K3$ surfaces $X_{[3,3,3]}$, $X_{[3,2,3]}$ and $X_{[3,0,3]}$.

1. Introduction

In the study of the geometry, arithmetic and moduli of $K3$ surfaces, elliptic $K3$ surfaces with large Picard number play a vital role. In 1977 Shioda and Inose [6] gave a

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https://doi.org/10.1016/j.jnt.2018.03.001

classification of singular $K3$ surfaces, that is, $K3$ surfaces with maximum Picard number. For this purpose, they constructed elliptic $K3$ surfaces $\mathcal{E}$ with two singular fibers of type II^* starting from the Kummer surface $\mathrm{Km}(E_1 \times E_2)$ with the product of two elliptic curves E_1 and E_2. They constructed $\mathcal{E}$ as a double cover of $\mathrm{Km}(E_1 \times E_2)$ with certain properties (now called a Shioda–Inose structure). Later, Inose [1] gave an explicit model of such an elliptic $K3$ surface as a quartic surface in $\mathbf{P}^3$, and remarked that it is the quotient of $\mathrm{Km}(E_1 \times E_2)$ by an involution. We call the Kodaira–Néron model of $\mathcal{E}$ the Inose surface associated with E_1 and E_2, and denote it by $\mathrm{Ino}(E_1, E_2)$. We thus have a "Kummer sandwich" diagram:

$$\mathrm{Km}(E_1 \times E_2) \overset{\pi_2}{\dashrightarrow} \mathrm{Ino}(E_1, E_2) \overset{\pi_1}{\dashrightarrow} \mathrm{Km}(E_1 \times E_2)$$

(cf. [7]). Also, $\mathcal{E}$ as an elliptic surface with two II^* fibers is denoted by $F^{(1)}_{E_1,E_2}$. This notation reflects that it is a part of the construction of elliptic $K3$ surfaces of high rank by the first named author [4], where he constructed $F^{(n)}_{E_1,E_2}$, $n = 1, \dots, 6$, which has various Mordell–Weil rank up to 18.

The structure of the Mordell–Weil lattice of $F^{(1)}_{E_1,E_2}$ is known to be isomorphic to $\mathrm{Hom}(E_1, E_2)\langle 2\rangle$ if E_1 and E_2 are nonisomorphic (see [8]). Here, for a lattice L, we denote by $L\langle n\rangle$ the lattice structure on L with the pairing multiplied by n. However, given an isogeny $\varphi \in \mathrm{Hom}(E_1, E_2)$ and the Weierstrass equation of $F^{(1)}_{E_1,E_2}$, it is quite difficult to write down the coordinates of the section corresponding to φ, and it has been worked out only in limited cases (cf. [9], [2]). Most known examples fall into the case where the degree of isogeny φ equals 2, in which case the calculations are straight forward. One particular example of the case $\deg \varphi = 4$ is dealt in [2, Example 9.2]. In this paper we consider a family of the pairs of elliptic curves E_1 and E_2 with an isogeny $\varphi : E_1 \to E_2$ of degree 3 defined over k. We write down a formula of the section of $F^{(1)}_{E_1,E_2}$ coming from φ defined over the base field k. To do so, we first work with the surface $F^{(6)}_{E_1,E_2}$, which has a simple affine model that can be viewed as a family of cubic curves with a rational point over k. We modify the method in [2] to find sections of $F^{(1)}_{E_1,E_2}$. We also give a section of $F^{(2)}_{E_1,E_2}$ coming from the isogeny φ, and give a basis defined over the field $k(E_1[2], E_2[2])$ when E_1 and E_2 do not have a complex multiplication.

In §7 we study some examples of singular $K3$ surfaces in detail. In particular, we determine a basis of the MWL of the Inose surface $F^{(1)}_{E_1,E_2}$ and that of $F^{(2)}_{E_1,E_2}$ for the singular $K3$ surfaces $X_{[3,3,3]}$, $X_{[3,2,3]}$ and $X_{[3,0,3]}$ which correspond to the quadratic forms $3x^2 + 3xy + 3y^2$, $3x^2 + 2xy + 3y^2$, and $3x^2 + 3y^2$ respectively.

Acknowledgments. We would like to express sincere gratitude to Professor Ichiro Shimada for useful discussions. We would also like to thank Professor Hisanori Ohashi for his helpful comments. The computer algebra system `Maple` was used in the calculations for this paper. Kuwata was partially supported by JSPS KAKENHI Grant Number JP26400023, and by the Chuo University Grant for Special Research. Utsumi was partially supported by the Ritsumeikan University Research Promotion Program for Aquiring Grants in-Aid for Scientific Research.