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On the mod p kernel of the theta operator and Eisenstein series

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Siegel modular forms in the space of the mod p kernel of the theta operator are constructed by the Eisenstein series in some odd-degree cases. Additionally, a similar result in the case of Hermitian modular forms is given.

1. Introduction

The theta operator is a kind of differential operator operating on modular forms. Let F be a Siegel modular form with the generalized q-expansion $F = \sum a(T)q^T$, $q^T := \exp(2\pi i \mathrm{tr}(TZ)))$. The theta operator Θ is defined as

$$\Theta : F = \sum a(T)q^T \longmapsto \Theta(F) := \sum a(T) \cdot \det(T)q^T,$$

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which is a generalization of the classical Ramanujan's θ-operator. It is known that the notion of singular modular form F is characterized by $\Theta(F) = 0$.

For a prime number p, the mod p kernel of the theta operator is defined as the set of modular form F such that $\Theta(F) \equiv 0 \pmod{p}$. Namely, the element in the kernel of the theta operator can be interpreted as a mod p analogue of the singular modular form.

In the case of Siegel modular forms of even degree, several examples are known (cf. Remark 2.3). In [11], the first author constructed such a form by using Siegel Eisenstein series in the case of even degree. However little is known about the existence of such a modular form in the case of odd degree.

In this paper, we shall show that some odd-degree Siegel Eisenstein series give examples of modular forms in the mod p kernel of the theta operator. Our proof is based on Katsurada's functional equation of Kitaoka's polynomial appearing as the main factor of the Siegel series.

Additionally, we give a similar result in the case of Hermitian modular forms. In this case, we use Ikeda's functional equation which is the corresponding result of Katsurada's one.

2. Siegel modular case

2.1. Siegel modular forms

Let $\Gamma^{(n)} = \mathrm{Sp}_n(\mathbb{Z})$ be the Siegel modular group of degree n and $M_k(\Gamma^{(n)})$ be the space of Siegel modular forms of weight k for $\Gamma^{(n)}$. Any element F in $M_k(\Gamma^{(n)})$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \le T \in \Lambda_n} a(T;F)q^T, \quad q^T := \exp(2\pi i \mathrm{tr}(TZ)), \quad Z \in \mathbb{H}_n,$$

where

$$\mathbb{H}_n = \{\, Z \in \mathrm{Sym}_n(\mathbb{C}) \mid \mathrm{Im}(Z) > 0 \,\} \text{ (the Siegel upper half space)},$$
$$\Lambda_n := \{\, T = (t_{jl}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{jj} \in \mathbb{Z},\, 2t_{jl} \in \mathbb{Z} \,\}.$$

We also denote by $S_k(\Gamma^{(n)})$ the space of $M_k(\Gamma^{(n)})$ consisting of cusp forms.

For a subring $R \subset \mathbb{C}$, $M_k(\Gamma^{(n)})_R$ (resp. $S_k(\Gamma^{(n)})_R$) consists of an element F in $M_k(\Gamma^{(n)})$ (resp. $S_k(\Gamma^{(n)})$) whose Fourier coefficients $a(T;F)$ lie in R.

2.2. Theta operator

For an element F in $M_k(\Gamma^{(n)})$, we define

$$\Theta : F = \sum a(T;F)q^T \longmapsto \Theta(F) := \sum a(T;F) \cdot \det(T) q^T$$

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