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THREE RAMANUJAN CONTINUED FRACTIONS WITH MODULARITY

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ABSTRACT. We study three Ramanujan continued fractions $c(\tau), W(\tau)$ and $T(\tau)$. In fact, $c(\tau)$ and $W(\tau)$ are modular functions of level 16, and $T(\tau)$ is a modular function of level 32. We first prove that the values of $c(\tau)$ and $W(\tau)$ can generate the ray class field modulo 4 over an imaginary quadratic field K. We also prove that $2/(1-c(\tau))$, $1/W(\tau)$, $T(\tau)+1/T(\tau)$ are algebraic integers for any imaginary quadratic quantity τ. Furthermore, we find the modular equations of $c(\tau), T(\tau)$ and $W(\tau)$ for any level, and we show that $c(\tau)$ and $W(\tau)$ satisfy the Kronecker's congruence. We can express the value $c(r\tau)$ (respectively, $T(r\tau), W(r\tau)$) in terms of radicals for any positive rational number r when the value $c(\tau)$ (respectively, $T(\tau), W(\tau)$) can be written as radicals.

1. INTRODUCTION

The *Rogers-Ramanujan continued fraction* $r(\tau)$ is a continued fraction which is written in two letters by Ramanujan to Hardy [16, pp. xxvii, xxviii], and it is defined by

$$r(\tau) = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}} \qquad (q := e^{2\pi i\tau}).$$

Ramanujan's letters contain several theorems including the evaluation of $r(\tau)$ and the relations between $r(\tau)$ and $r(n\tau)$ $(n=2,\ldots,5)$. These amazing results are recently proved by the theory of modular forms [3, 6, 7] because $r(\tau)$ is a generator of the field of modular functions on $\Gamma(5)$.

There are several works which are motivated by Ramanujan's work. Among them, we introduce the following three continued fractions $c(\tau), T(\tau)$ and $W(\tau)$:

$$c(\tau) = \cfrac{1}{1+\cfrac{2q}{1-q^2+\cfrac{q^2(1+q^2)^2}{1-q^6+\cfrac{q^4(1+q^4)^2}{1-q^{10}+\cdots}}}},$$

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