Journal of Number Theory 188 (2018) 357–370

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the density of the odd values of the partition function, II: An infinite conjectural framework

Samuel D. Judge, Fabrizio Zanello *

Department of Mathematical Sciences, Michigan Tech, Houghton, MI 49931, United States

A R T I C L E   I N F O

Article history:
Received 11 November 2017
Accepted 13 January 2018
Available online 19 February 2018
Communicated by L. Smajlovic

MSC:
primary 11P83
secondary 05A17, 11P84, 11P82, 11F33

Keywords:
Partition function
Multipartition function
Density odd values
Partition identity
Modular form modulo 2

A B S T R A C T

We continue our study of a basic but seemingly intractable problem in integer partition theory, namely the conjecture that $p(n)$ is odd exactly 50% of the time. Here, we greatly extend on our previous paper by providing a doubly-indexed, infinite framework of conjectural identities modulo 2, and show how to, in principle, prove each such identity. However, our conjecture remains open in full generality.

A striking consequence is that, under suitable existence conditions, if any t-multipartition function is odd with positive density and $t \not\equiv 0 \pmod 3$, then $p(n)$ is also odd with positive density. These are all facts that appear virtually impossible to show unconditionally today.

Our arguments employ a combination of algebraic and analytic methods, including certain technical tools recently developed by Radu in his study of the parity of the Fourier coefficients of modular forms.

* Corresponding author.
E-mail addresses: sdjudge@mtu.edu (S.D. Judge), zanello@mit.edu, zanello@mtu.edu (F. Zanello).

https://doi.org/10.1016/j.jnt.2018.01.016
0022-314X/

1. Introduction

Let $p(n)$ denote the number of *partitions* of a nonnegative integer n, where a partition of n is an unordered set of positive integers that sum to n. It is well known that the generating function of $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\prod_{i=1}^{\infty}(1-q^i)}.$$

For any positive integer t, define $p_t(n)$ by

$$\sum_{n=0}^{\infty} p_t(n)q^n = \frac{1}{\prod_{i=1}^{\infty}(1-q^i)^t}.$$

Hence, $p_t(n)$ denotes the number of t-*multipartitions* (also called t-*colored partitions*) of n. In particular, $p_1(n) = p(n)$.

The goal of this paper is to provide additional insight into the long-standing question of estimating the density of the odd values of the partition function. As Paul Monsky once pointedly stated [22], "the best minds of our generation haven't gotten anywhere with understanding the parity of $p(n)$." It is a famous conjecture of Parkin and Shanks [30] that the even and the odd values of $p(n)$ are equidistributed (see also Section 4 of [26]). Currently, the best result, which has improved on the work of multiple authors (see for example [1,11,14,25,29]), is due to Bellaïche, Green, and Soundararajan [10] and states that the number of odd values of $p(n)$ for $n \leq x$ has at least the order of $\frac{\sqrt{x}}{\log\log x}$, for $x \to \infty$. In fact, their bound holds for any t-multipartition function $p_t(n)$, thus improving on a result of the second author (cf. [37], which includes an elementary proof for $t = 3$). We note that the current record lower bound on the number of *even* values of $p(n)$ has the order of $\sqrt{x}\log\log x$ ([11]. Unlike in the odd case, a lower bound of $\sqrt{x}$ for the even values is trivial).

We define the *density* of the odd values of $p_t(n)$ in the natural way:

$$\delta_t = \lim_{x\to\infty} \frac{\#\{n \leq x : p_t(n) \text{ is odd}\}}{x},$$

if the limit exists. Note that, for obvious parity reasons, it suffices to restrict our attention to the case of t odd. Given the above bounds, we are still very far from showing that $\delta_t > 0$, for any t. In fact, it is not even known at this time that any δ_t exists.

In [17], in collaboration with Keith, we generalized the conjecture of Parkin and Shanks (i.e., $\delta_1 = 1/2$) to every $t \geq 1$. Namely:

Conjecture 1.1 *([17], Conjecture 1). The density δ_t exists and equals $1/2$, for any positive odd integer t. Equivalently, if $t = 2^k t_0$ with $t_0 \geq 1$ odd, then δ_t exists and equals 2^{-k-1}.*