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Fine Selmer groups of congruent Galois representations

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A B S T R A C T

In this paper, we study the fine Selmer groups of two congruent Galois representations over an admissible p-adic Lie extension. We show that under appropriate congruence conditions, if the dual fine Selmer group of one is pseudo-null, so is the other. Our results also compare the π-primary submodules of the two dual fine Selmer groups. We then apply our results to compare the structure of Galois group of the maximal abelian unramified pro-p extension of an admissible p-adic Lie extension and the structure of the dual fine Selmer group over the said admissible p-adic Lie extension.

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1. Introduction

Throughout the paper, p will always denote a fixed prime. Let F be a number field. If $p = 2$, we assume that F is totally imaginary. We write F^{cyc} for the cyclotomic $\mathbb{Z}_p$-extension of F, whose Galois group $\mathrm{Gal}(F^{\mathrm{cyc}}/F)$ is in turn denoted Γ. Denote by $K(F^{\mathrm{cyc}})$ the maximal abelian unramified pro-p extension of F^{cyc} in which every prime of F^{cyc} above p splits completely. In [19], Iwasawa proved that $\mathrm{Gal}(K(F^{\mathrm{cyc}})/F^{\mathrm{cyc}})$ is a finitely generated torsion $\mathbb{Z}_p[\![\Gamma]\!]$-module, and he further conjectured that this Galois group is finitely generated over $\mathbb{Z}_p$ (see also [18]). Throughout this article, we shall call this conjecture the *Iwasawa μ-conjecture*. On the other hand, it has been known that this finite generation property *does not* hold for the dual of the classical Selmer group of an abelian variety over the cyclotomic $\mathbb{Z}_p$-extension in general (see [28, §10, Example 2]). It was only about a decade ago that Coates and the second named author [7,37] gave a correct formulation of the analogue of the Iwasawa μ-conjecture for an elliptic curve. Namely, they considered a smaller group, called the fine Selmer group, which is a subgroup of the classical Selmer group, and they conjectured that the Pontryagin dual of this fine Selmer group over F^{cyc} is finitely generated over $\mathbb{Z}_p$ [7, Conjecture A]. Since then, analogues of this conjecture have been formulated for fine Selmer groups attached to more general Galois representations (see [2,20,21,25,27]). In this paper, we shall collectively (and loosely) address these conjectures as Conjecture A. A striking observation is that, besides being a natural analogue of the Iwasawa μ-conjecture, Conjecture A is related to the latter conjecture in a very precise manner (see [7, Theorem 3.4], [25, Theorem 3.5], [27, Theorem 5.5], [37, Theorem 4.5] or [40, Section 8]; also see Theorem 3.1 below).

In their paper [7], Coates and the second author also studied the structure of the fine Selmer group over extensions of F whose Galois group $G = \mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group of dimension larger than 1. There they formulated an important conjecture on the structure of the Pontryagin dual of the fine Selmer group of an abelian variety which predicts that the said module is pseudo-null over the Iwasawa algebra $\mathbb{Z}_p[\![G]\!]$ (see [7, Conjecture B]). To some extent, their conjecture can be thought as an analogue of a conjecture of Greenberg, which we now briefly describe. Recall that a Galois extension F_∞ of F is said to be a strongly admissible, pro-p, p-adic Lie extension of F if (i) $G = \mathrm{Gal}(F_\infty/F)$ is a compact pro-p, p-adic Lie group without p-torsion, (ii) F_∞ contains the cyclotomic $\mathbb{Z}_p$ extension F^{cyc} of F and (iii) F_∞ is unramified outside a finite set of primes. We denote by H the Galois group $\mathrm{Gal}(F_\infty/F^{\mathrm{cyc}})$. Let $K(F_\infty)$ denote the maximal abelian unramified pro-p extension of F_∞ in which every prime above p splits completely. When F_∞ is the composite of all the $\mathbb{Z}_p$-extensions of F, Greenberg [12] conjectured that $\mathrm{Gal}(K(F_\infty)/F_\infty)$ is pseudo-null over $\mathbb{Z}_p[\![G]\!]$. (Actually, to be more precise, Greenberg's original conjecture is concerned with the pseudo-nullity of a slightly bigger Galois group.) For a general F_∞, the validity of the pseudo-nullity of $\mathrm{Gal}(K(F_\infty)/F_\infty)$ is not guaranteed, and this was first observed by Hachimori and Sharifi in [14], where they constructed a

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