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On the equation $A!B! = C!$ ☆

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ABSTRACT

We consider the equation in the title in positive integers A, B, C. We give an explicit upper bound for C in terms of the difference $k := B - A$. Further, we show that for $k \leq 10^6$ this equation has only one (long known) non-trivial solution, given by $6!7! = 10!$.

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1. Introduction and the main result

The question of finding all products of factorials yielding a factorial is a long standing problem, studied by many authors. Here we only mention a few related results; for a survey of the topic see e.g. Guy [8], section B23. Consider the equation

$$n! = \prod_{i=1}^{r} a_i! \tag{1}$$

with $r \geq 2$ in positive integers $n, a_1, \ldots, a_r$, with $a_1 \geq \cdots \geq a_r > 1$. Observe that this equation has infinitely many solutions given by

$$n = a_2! \ldots a_r!,\ a_1 = n - 1,\ \text{with } a_2, \ldots, a_r \text{ arbitrary.}$$

For example, we have $6! = 5!3!$ or $12! = 11!3!2!$. Such solutions are called trivial. Obviously, equation (1) has infinitely many trivial solutions. On the other hand, according to a conjecture of Surányi, the only non-trivial solution to (1) with $r = 2$ is $10! = 7!6!$, while a conjecture of Hickerson predicts that the only non-trivial solutions to (1) are given by $9! = 7!3!3!2!$, $10! = 7!6! = 7!5!3!$, $16! = 14!5!2!$ (see e.g. Erdős [4], pp. 27–28). These conjectures have been checked for $n \leq 10^6$ by Caldwell [3]. Erdős [4] (see Theorem 2) proved that writing $P(m)$ for the largest prime factor of the positive integer m (with the convention $P(1) = 1$), the assertion

$$P(n(n + 1)) > 4 \log n \tag{2}$$

would imply that equation (1) has only finitely many non-trivial solutions – however, (2) is far from being established. (See also [7], p. 70.) Luca [10] proved that assuming the *abc*-conjecture, (1) has only finitely may solutions. This result (beside obtaining other related theorems) has been made more explicit by Luca, Saradha and Shorey [11].

We also mention that after multiplying both sides of (1) by $n!$, we get an equation of the form

$$n! \prod_{i=1}^{r} a_i! = y^2.$$

This equation also attracted a lot of attention. For related results, here we only mention a classical paper of Erdős and Graham [6] together with the recent paper of Luca, Saradha and Shorey [11], and the references there.

In this paper we consider the case $r = 2$, and rewrite equation (1) as

$$A!B! = C! \tag{3}$$

with positive integers A, B, C satisfying $C \geq B \geq A > 1$.