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Congruences modulo powers of 5 for k-colored partitions

Dazhao Tang

College of Mathematics and Statistics, Chongqing University, Huxi Campus LD206, Chongqing 401331, PR China

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A B S T R A C T

Let $p_{-k}(n)$ enumerate the number of k-colored partitions of n. In this paper, we establish some infinite families of congruences modulo 25 for k-colored partitions. Furthermore, we prove some infinite families of Ramanujan-type congruences modulo powers of 5 for $p_{-k}(n)$ with $k = 2, 6$, and 7. For example, for all integers $n \geq 0$ and $\alpha \geq 1$, we prove that

$$p_{-2}\left(5^{2\alpha-1}n + \frac{7 \times 5^{2\alpha-1}+1}{12}\right) \equiv 0 \pmod{5^{\alpha}}$$

and

$$p_{-2}\left(5^{2\alpha}n + \frac{11 \times 5^{2\alpha}+1}{12}\right) \equiv 0 \pmod{5^{\alpha+1}}.$$

E-mail address: dazhaotang@sina.com.

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1. Introduction

A *partition* [1] of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r > 0$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i's are called the *parts* of the partition. Let $p(n)$ denote the number of partitions of n, then

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Here and throughout the paper, we adopt the following customary notation on partitions and q-series:

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1.$$

A partition is called a k-colored partition if each part can appear as k colors. Let $p_{-k}(n)$ count the number of k-colored partitions of n. The generating function of $p_{-k}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{-k}(n)q^n = \frac{1}{(q;q)_\infty^k}.$$

For convention, we denote $p_{-1}(n) = p(n)$.

Many congruences modulo 5 and 25 enjoyed by $p_{-k}(n)$ have been found. For example, Ramanujan's so-called "most beautiful identity" for partition function $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.1}$$

which readily implies one of his three classical partition congruences, namely,

$$p(5n+4) \equiv 0 \pmod{5}. \tag{1.2}$$

Further, we have, modulo 25,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} \equiv 5\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty}$$
$$= 5(q^5;q^5)_\infty^4 \sum_{n=0}^{\infty} p(n)q^n,$$

from which it follows easily from (1.2) that

$$p(25n+24) \equiv 0 \pmod{25}.$$

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