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Characterization of uniformly distributed sets and maximal density sets

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ABSTRACT

We consider different kinds of densities on natural numbers and investigate the problems on densities stated in the paper "Open problems on densities II" by Giuliano, Grekos and Mišík. We find answers to several of those problems and obtain new results related to the notions mentioned in that paper like distribution functions and density sets.

1. Introduction

In this paper we will work with densities of subsets of natural numbers. We will denote the set of natural numbers as $\mathbb{N} = \{1, 2 \ldots\}$. The general notion of density describes a function defined on $\mathcal{P}(\mathbb{N})$ (or only a part of it) such that $f(\emptyset) = 0$, $f(\mathbb{N}) = 1$ and $f(A) \leq f(B)$ for all $A, B \subseteq \mathbb{N}$ such that $A \subseteq B$ and $f(A), f(B)$ are defined. Instead of

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investigating the general definition of density, we will focus on several specific types of densities that we introduce later in this section.

Throughout the paper, we will use the following notation for $A \subseteq \mathbb{N}$ and $n, m \in \mathbb{R}$, $n \leq m$:

$$A(n) = |A \cap [1, n]|, \ A(n, m) = |A \cap [n, m]|.$$

On the sequel, even if n and m are not natural numbers, we will treat any interval $[n, m]$ as the interval on integers $[\lceil n \rceil, \lfloor m \rfloor] \cap \mathbb{N}$.

For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $d_n(A) = A(n)/n$. We will define *upper* and *lower asymptotic densities* of a set $A \subseteq \mathbb{N}$, respectively as

$$\overline{d}(A) = \limsup_{n\to\infty} d_n(A),$$

$$\underline{d}(A) = \liminf_{n\to\infty} d_n(A).$$

When $\overline{d}(A) = \underline{d}(A)$, then we say that A has an *asymptotic density* and we denote it by $d(A)$.

The following kind of density was studied by Pólya in [8].

$$\overline{\overline{d}}(A) = \lim_{\theta\to 1^-} \limsup_{n\to\infty} \frac{A(n) - A(\theta n)}{(1-\theta)n},$$

$$\underline{\underline{d}}(A) = \lim_{\theta\to 1^-} \liminf_{n\to\infty} \frac{A(n) - A(\theta n)}{(1-\theta)n},$$

where he calls them *maximal* and *minimal* density of a set $A \subseteq \mathbb{N}$, respectively.

It can be seen [8, Satz 8] that

$$\underline{\underline{d}}(A) = \sup\{d(B) : B \subseteq A\}, \ \overline{\overline{d}}(A) = \inf\{d(B) : B \supseteq A\}.$$

It is clear that $\underline{\underline{d}}(A) \leq \underline{d}(A) \leq \overline{d}(A) \leq \overline{\overline{d}}(A)$.

Now we will present the definitions of several classical and well known kinds of densities.

We define the *upper uniform density* of a set $A \subseteq \mathbb{N}$ as

$$\overline{u}(A) = \lim_{h\to\infty} \limsup_{n\to\infty} \frac{A[n, n+h]}{\mathbb{N}[n, n+h]}$$

and its *lower uniform density* as

$$\underline{u}(A) = \lim_{h\to\infty} \liminf_{n\to\infty} \frac{A[n, n+h]}{\mathbb{N}[n, n+h]}.$$

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