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Galois groups in a family of dynatomic polynomials

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ABSTRACT

For every nonconstant polynomial $f \in \mathbb{Q}[x]$, let $\Phi_{4,f}$ denote the fourth dynatomic polynomial of f. We determine here the structure of the Galois group and the degrees of the irreducible factors of $\Phi_{4,f}$ for every quadratic polynomial f. As an application we prove new results related to a uniform boundedness conjecture of Morton and Silverman. In particular we show that if f is a quadratic polynomial, then, for more than 39% of all primes p, f does not have a point of period four in $\mathbb{Q}_p$.

1. Introduction

Let $f \in \mathbb{Q}[x]$ be a nonconstant polynomial. For every positive integer n, let f^n denote the n-fold composition of f with itself. An algebraic number α is called *periodic* under iteration of f if there exists $n \geq 1$ such that $f^n(\alpha) = \alpha$; in that case the least such n is called the *period* of α.

A fundamental conjecture of Morton and Silverman [23] would imply that as f varies over all polynomials of fixed degree $d > 1$, the possible periods of rational numbers under

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iteration of f remain bounded. In [26] Poonen studied the case $d = 2$ and conjectured that no quadratic polynomial over $\mathbb{Q}$ has a rational point of period greater than 3.

A useful construction for studying these conjectures is that of a dynatomic polynomial. For every nonconstant polynomial $f \in \mathbb{Q}[x]$ and every positive integer n, the nth *dynatomic polynomial* of f is defined by the equation

$$\Phi_{n,f}(x) = \prod_{d|n} \left(f^d(x) - x\right)^{\mu(n/d)}, \tag{1.1}$$

where μ is the Möbius function. The key property that motivates this definition is that the roots of $\Phi_{n,f}$ are precisely the algebraic numbers having period n under iteration of f, except in rare cases when some roots may have period smaller than n; see [22, Thm. 2.4] for further details. By studying algebraic properties of the dynatomic polynomials of f one may thus hope to gain information about the dynamical properties of f as a map $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}$. With this in mind, let us now focus on the case of quadratic polynomials and discuss the following problem, which we consider to be especially important for understanding the dynamics of such maps.

Problem 1.1. Given $n \geq 1$, determine all possible groups that can arise as the Galois group of $\Phi_{n,f}$ for some quadratic polynomial $f \in \mathbb{Q}[x]$. Furthermore, determine all possible factorization types[1] of $\Phi_{n,f}$ that can arise as f varies over all quadratic polynomials.

This problem is easily solved for $n = 1$ and 2. The case $n = 3$ is substantially harder, but was solved by Morton [20]. The purpose of this article is to treat the case $n = 4$. At present there appears to be no published work concerning the structure of the Galois groups of the polynomials $\Phi_{n,f}$ in this case. Regarding factorization types we are only aware of two results in the literature: Morton [21, Thm. 4] showed that $\Phi_{4,f}$ can never have a factor of degree 1, and Panraksa [25, Thm. 2.3.5] showed that $\Phi_{4,f}$ cannot have four or more irreducible quadratic factors.

In order to state our results for $n = 4$ we introduce some notation. For every quadratic polynomial $f \in \mathbb{Q}[x]$ there exist a unique rational number c and a unique linear polynomial $l \in \mathbb{Q}[x]$ such that

$$l \circ f \circ l^{-1} = \phi_c(x) := x^2 + c.$$

The polynomials f and ϕ_c share all the properties we are concerned with in this article; in particular, their dynatomic polynomials factor in the same way and have the same Galois group (see [13, §2.2]). In stating our results we may therefore restrict attention to the family of polynomials $\{\phi_c(x) : c \in \mathbb{Q}\}$. To ease notation we will write $\Phi_{4,c}$ instead of Φ_{4,ϕ_c}.

[1] By the *factorization type* of a polynomial $F \in \mathbb{Q}[x]$ we mean the multiset of degrees of irreducible factors of F.

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