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Quantum Jacobi forms and balanced unimodal sequences ☆

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A B S T R A C T

The notion of a quantum Jacobi form was defined in 2016 by Bringmann and the second author in [1], marrying Zagier's notion of a quantum modular form [12] with that of a Jacobi form. Only one example of such a function has been given to-date (see [1]). Here, we prove that two combinatorial rank generating functions for certain balanced unimodal sequences, studied previously by Kim, Lim and Lovejoy [8], are also natural examples of quantum Jacobi forms. These two combinatorial functions are also duals to partial theta functions studied by Ramanujan. Additionally, we prove that these two functions have the stronger property that they exhibit mock Jacobi transformations in $\mathbb{C} \times \mathbb{H}$ as well as quantum Jacobi transformations in $\mathbb{Q} \times \mathbb{Q}$. As corollaries to these results, we use quantum Jacobi properties to establish new, simpler expressions for these functions as simple Laurent polynomials when evaluated at pairs of rational numbers.

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1. Introduction and statement of results

A sequence of integers $\{a_j\}_{j=1}^{s} (s \in \mathbb{N})$ is called a *strongly unimodal sequence of size* n if there exists a positive integer k such that

$$0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_s > 0,$$

and $a_1 + \cdots + a_s = n$. The *rank* of a strongly unimodal sequence is equal to $2k - s - 1$.[1] *Odd-balanced unimodal sequences* allow odd parts to repeat on either side of the peak a_k, but they must be identical on either side, and the peak must be even. Let $\nu(m, n)$ be the number of such sequences with rank m and size $2n$.[2] The two-variable rank generating function for odd-balanced unimodal sequences satisfies [8]

$$q\mathcal{V}(w; q) := \sum_{n\geq 0} \frac{(-wq; q)_n (-w^{-1}q; q)_n q^{n+1}}{(q; q^2)_{n+1}} = \sum_{\substack{n\geq 1\\ m\in\mathbb{Z}}} \nu(m, n) w^m q^n. \tag{1.1}$$

Here and throughout, the q-Pochhammer symbol is defined by $(w; q)_n := \prod_{j=0}^{n-1}(1 - wq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$.

In [1], the notion of a *quantum Jacobi form* is introduced by Bringmann and the second author, marrying Zagier's notion of a quantum modular form [12] with that of a Jacobi form (see §2 for a precise definition and more details). In [1] it is also proved that the two-variable combinatorial generating function for ranks of strongly unimodal sequences is an example of such a form – this is the first and only example of such a function in the literature to-date. Here we prove that two additional combinatorial functions are quantum Jacobi forms, one of which is a normalized version of the function in (1.1). Precisely, we define

$$\mathcal{V}^{+}(z; \tau) := 2\cos(\pi z)\mathcal{V}(w; q)q^{\frac{7}{8}}, \tag{1.2}$$

on $\mathbb{C} \times \mathbb{H}$ with $w := e(z), q := e(\tau)$, where here and throughout $e(\alpha) := e^{2\pi i\alpha}$. In Theorem 1.1, we prove that $\mathcal{V}^{+}$ is a quantum Jacobi form with respect to the congruence subgroup $\Gamma_0(4) \subseteq \mathrm{SL}_2(\mathbb{Z})$. In fact, we prove that $\mathcal{V}^{+}$ has the stronger property that it exhibits mock Jacobi transformations in $\mathbb{C}\times\mathbb{H}$ as well as quantum Jacobi transformations in $\mathbb{Q} \times \mathbb{Q}$. By exploiting these quantum Jacobi properties, we also obtain a new simple expression for $\mathcal{V}^{+}$ as a Laurent polynomial when evaluated at pairs of rational numbers (see Theorem 1.3).

In order to state our results related to $\mathcal{V}^{+}$ precisely, we first define the "errors of modularity"

[1] Here, we use the definition of rank as in [8]. Other sources such as [2] define rank to be $-(2k - s - 1) = s - 2k + 1$.
[2] Note that $\nu(m, n) = v(m, n - 1)$, where v is as defined in [8].

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