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Pair correlation of zeros of the real and imaginary parts of the Riemann zeta-function

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We show that if the Riemann Hypothesis is true for the Riemann zeta-function, $\zeta(s)$, and $0 < a < 1/2$, then all but a finite number of the zeros of $\Re\zeta(a+it)$, $\Im\zeta(a+it)$, and similar functions are simple. We also study the pair correlation of the zeros of these functions assuming the Riemann Hypothesis is true and $0 < a \le 1/2$.

1. Introduction and statement of results

Let $w = u + iv$ be a complex variable and let $\zeta(w)$ be the Riemann zeta-function. We assume the Riemann Hypothesis (RH) is true throughout unless otherwise indicated. Our goal is to investigate the distribution of the zeros of $\Re\zeta(a+iv)$ and $\Im\zeta(a+iv)$, when $0 < a \le 1/2$. The zeros of these two functions coincide with those of

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$$\zeta(a+w)+\zeta(a-w) \quad \text{and} \quad \zeta(a+w)-\zeta(a-w),$$

respectively, on the line $\Re w=0$. Since the latter are analytic except for simple poles at $w=1-a$ and $a-1$, we will work directly with these functions instead.

One can just as well investigate the zeros of the more general functions

$$f_a(w,\theta)=\zeta(a+w)+e^{i\theta}\zeta(a-w) \qquad (\theta\in[0,2\pi)).$$

Note that $f_a(w,0)=2\Re\,\zeta(a+w)$ and $f_a(w,\pi)=2i\Im\,\zeta(a+w)$ on the line $\Re w=0$. Note also that $f_a(w,\theta)$ satisfies the functional equation

$$\begin{aligned} f_a(-w,-\theta) &= \zeta(a-w)+e^{-i\theta}\zeta(a+w) \\ &= e^{-i\theta} f_a(w,\theta). \end{aligned}$$

When $a=1/2$ we see that

$$f_{1/2}(w,\theta)=\zeta(1/2+w)+e^{i\theta}\zeta(1/2-w)=\zeta(1/2+w)(1+e^{i\theta}\chi(1/2-w)),$$

where $\chi(w)$ is the factor from the functional equation

$$\zeta(w)=\chi(w)\zeta(1-w). \tag{1.1}$$

Thus, if RH is true, then iv is a zero of $f_{1/2}(w,\theta)$ on $\Re w=0$ if and only if either $\zeta(1/2+iv)=0$ or $\chi(1/2+iv)=-e^{i\theta}$.

From now on we assume that $\theta\in[0,2\pi)$ is fixed and write $f_a(w)$ for $f_a(w,\theta)$. Let $\rho_a=\beta_a+i\gamma_a$ denote a typical zero of $f_a(w)$. M. Z. Garaev [1] and H. Ki [2] have shown that when $\theta=0$ or π, $f_a(w)$ has

$$N_a(T)=\frac{T}{\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(\log T)$$

zeros with $0<\gamma_a\le T$. Although this result is for a fixed a, it is easy to see from their arguments that one may take the constant implied by the O-term to be absolute when $0<a\le 1/2$. Ki has also proved that if RH is true and $0<a\le 1/2$, all but a finite number of the nonreal zeros of $f_a(w)$ lie on the line $\Re w=0$. (He proved a similar result for $a\le 0$ without assuming RH.) Thus, there exists a real number T_a such that $\beta_a=0$ when $\gamma_a>T_a$. Here too, an inspection of the proof reveals that there exists a uniform lower bound T_0 that works for all $a\in(0,1/2]$. Moreover, it is clear that only slight changes are needed to establish the corresponding results for other values of θ.

When $\theta=0$ or π the functions $f_a(w)$ have "trivial" real zeros as well. M. Z. Garaev [1] showed that for each a there exists a number $U_0>0$ such that every zero of $f_a(w)$ outside the strip $|\Re w|\le U_0$ is real, and that there is exactly one in each interval $(2n-1+a,2n+1+a)$. Garaev's U_0 depends on a, but once again one sees from the proof that it may be chosen independently of a when $0<a\le 1/2$. One can easily show