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Popular differences and generalized Sidon sets

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ABSTRACT

For a subset $A \subseteq [N]$, we define the representation function $r_{A-A}(d) := \#\{(a, a') \in A \times A : d = a - a'\}$ and define $M_D(A) := \max_{1 \leq d < D} r_{A-A}(d)$ for $D > 1$. We study the smallest possible value of $M_D(A)$ as A ranges over all possible subsets of $[N]$ with a given size. We give explicit asymptotic expressions with constant coefficients determined for a large range of D. We shall also see how this problem connects to a well-known problem about generalized Sidon sets.

1. Introduction

For a positive integer N, denote by $[N]$ the discrete interval $\{1, 2, \cdots, N\}$. For a subset $A \subseteq [N]$, we study the difference set

$$A - A := \{a - a' : a, a' \in A\}.$$

We introduce the representation function r_{A-A}, defined by

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$$r_{A-A}(d) := \#\{(a, a') \in A \times A : d = a - a'\}.$$

The representation function has been investigated a lot in the literature. See, for example, [2], [3]. In this paper, we are interested in the maximum size of $r_{A-A}(d)$, as d ranges over all positive integers smaller than D for different thresholds D. More precisely, let

$$M_D(A) := \max_{1 \leq d < D} r_{A-A}(d)$$

and $f_D(N, \alpha)$ be the smallest possible value of $M_D(A)$, as A ranges over all subsets $A \subseteq [N]$ with $|A| \geq \alpha N$, i.e.,

$$f_D(N, \alpha) := \min_{A \subseteq [N], |A| \geq \alpha N} M_D(A).$$

Theorem 1.1. *Let $2 \leq D \leq N$ be positive integers and let $\alpha \in (0, 1)$. Write $D = (1+\gamma)\alpha^{-1}$ for some $-1 < \gamma$, then we have the following:*

(1) *If $\gamma \leq 0$, then $f_D(N, \alpha) = 0$.*
(2) *If $0 < \gamma \leq 1 - \delta$ for some positive number δ, $N \gg_\delta \alpha^{-2}$ and D is sufficiently large in terms of δ^{-1}, then $f_D(N, \alpha) = (1 + o(1))\frac{2\gamma}{(1+\gamma)^2}\alpha^2 N$.*
(3) *If $\alpha^2 N \to \infty$, $\gamma \to \infty$ and $D\alpha \to \infty$ as $N \to \infty$, then $f_D(N, \alpha) = (1 + o(1))\alpha^2 N$.*

Here $\gamma = \gamma(N), \alpha = \alpha(N), D = D(N)$. They are all regarded as functions of N.

Remark 1.2. 1) When $\alpha \geq 1/2$, the above conclusions can be easily checked. So we focus on $\alpha \in (0, 1/2)$ from now on.
2) A more precise statement of Theorem 1.1 (2) is Theorem 2.1.

1.1. Connections with Sidon sets

We shall see that there are some connections between problems about Sidon sets and the problem we are investigating in the case $D = N$.

Definition 1.1. (Sidon sets and g-Sidon sets) A set of natural numbers S is called a Sidon set if the equation $a+b = c+d$ has only the trivial solution $\{a, b\} = \{c, d\}$, where a, b, c, d are elements of the set S. A set A is called g-Sidon set if for any integer x we have:

$$r_{A+A}(x) := \#\{(a, b) \in A \times A : a + b = x\} \leq g.$$

Cilleruelo, Ruzsa and Vinuesa (2010) [1] proved the following result.

Theorem 1.3. *Define $\beta_g(N) := \max\{|A|\}$ where $A \subseteq [N]$ and A is a g-Sidon set. Then,*

$$\sigma_1(g)\sqrt{gN}(1 - o(1)) \leq \beta_g(N) \leq \sigma_2(g)\sqrt{gN}(1 + o(1)).$$

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