ARTICLE IN PRESS YJNTH:5890

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Conditional expanding bounds for two-variable functions over arbitrary fields

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ARTICLE INFO

Article history:
Received 6 June 2017
Received in revised form 26 September 2017
Accepted 26 September 2017
Available online xxxx
Communicated by L. Smajlovic

Keywords:
Sum-product estimates
Incidence geometry
Expanders

ABSTRACT

In this paper, we prove some results on the sum-product problem over arbitrary fields which improve and generalize results given by Hegyvári and Hennecart [5]. More precisely, we prove that, for related pairs of two-variable functions $f(x, y)$ and $g(x, y)$, if A and B are two sets in an arbitrary field $\mathbb{F}$ with $|A| = |B|$, then

$$\max\{|f(A, B)|, |g(A, B)|\} \gg |A|^{1+c},$$

for some $c > 0$.

1. Introduction

Throughout this paper, by $\mathbb{F}$ we refer to any arbitrary field, while by $\mathbb{F}_p$, we only refer to the fields of prime order p. We denote the set of non-zero elements by $\mathbb{F}^*$ and $\mathbb{F}_p^*$, respectively. Furthermore, we use the following convention: if the characteristic of $\mathbb{F}$ is positive, then we denote its characteristic by p; if the characteristic of $\mathbb{F}$ is zero, then we

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https://doi.org/10.1016/j.jnt.2017.09.020
0022-314X/

Please cite this article in press as: H. Nassajian Mojarrad, T. Pham, Conditional expanding bounds for two-variable functions over arbitrary fields, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.09.020

set $p = \infty$. So a term like $N < p^{5/8}$ is restrictive in positive characteristic, but vacuous for zero one.

For $A \subset \mathbb{F}$, the sum and the product sets are defined as follows:

$$A + A = \{a + a' : a, a' \in A\},\ A \cdot A = \{a \cdot a' : a, a' \in A\}.$$

For $A \subset \mathbb{F}_p$, Bourgain, Katz and Tao ([2]) proved that if $p^\delta < |A| < p^{1-\delta}$ for some $\delta > 0$, then we have

$$\max\{|A + A|, |A \cdot A|\} \gg |A|^{1+\epsilon},$$

for some $\epsilon = \epsilon(\delta) > 0$. Here, and throughout, by $X \ll Y$ we mean that there exists the constant $C > 0$ such that $X \le CY$.

In a recent paper [8], Roche-Newton, Rudnev, and Shkredov improved and generalized this result to the setting of arbitrary fields. More precisely, they showed that for $A \subset \mathbb{F}$, the sum set, product set, and quotient set satisfy

$$\max\{|A \pm A|, |A \cdot A|\} \gg |A|^{6/5},\ \ \max\{|A \pm A|, |A : A|\} \gg |A|^{6/5}.$$

We note that the same bound also holds for $|A(1 + A)|$ [10], and $|A + A^2|$, $\max\{|A + A|, |A^2 + A^2|\}$ [7]. We refer the reader to [1,3,8,6] and references therein for recent results on the sum-product topic.

Let G be a subgroup of $\mathbb{F}^*$, and $g \colon G \to \mathbb{F}^*$ be an arbitrary function. We define

$$\mu(g) = \max_{t \in \mathbb{F}^*} |\{x \in G \colon g(x) = t\}|.$$

For $A, B \subset \mathbb{F}_p$ and a related pair of two-variable functions $f(x, y)$ and $g(x, y)$ in $\mathbb{F}_p[x, y]$, Hegyvári and Hennecart [5], using graph theoretic techniques, proved that there exists $\Delta(\alpha) > 0$ such that for $|A| = |B| = p^\alpha$, we have

$$\max\{|f(A, B)|, |g(A, B)|\} \gg |A|^{1+\Delta(\alpha)},$$

where $f(A, B) = \{f(a, b) \colon a \in A, b \in B\}$. More precisely, they established the following results.

Theorem 1.1 *(**Hegyvári and Hennecart**, [5]). Let G be a subgroup of $\mathbb{F}_p^*$. Consider the function $f(x, y) = g(x)(h(x) + y)$ on $G \times \mathbb{F}_p^*$, where $g, h \colon G \to \mathbb{F}_p^*$ are arbitrary functions. Define $m = \mu(g \cdot h)$. For any subsets $A \subset G$ and $B, C \subset \mathbb{F}_p^*$, we have*

$$|f(A, B)|\,|B \cdot C| \gg \min\left\{\frac{|A||B|^2|C|}{pm^2}, \frac{p|B|}{m}\right\}.$$

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