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On the number of distinct prime factors of a sum of super-powers ☆

Paolo Leonetti[a], Salvatore Tringali[b,c,*]

[a] *Department of Decision Sciences, Università L. Bocconi, via Roentgen 1, 20136 Milano, Italy*
[b] *Department of Mathematics, Texas A&M University at Qatar, PO Box 23874, Doha, Qatar*
[c] *Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstr. 36, 8010 Graz, Austria*

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Given $k, \ell \in \mathbf{N}^+$, let $x_{i,j}$ be, for $1 \le i \le k$ and $0 \le j \le \ell$, some fixed integers, and define, for every $n \in \mathbf{N}^+$, $s_n := \sum_{i=1}^k \prod_{j=0}^\ell x_{i,j}^{n^j}$. We prove that the following are equivalent:

(a) There are a real $\theta > 1$ and infinitely many indices n for which the number of distinct prime factors of s_n is greater than the super-logarithm of n to base θ.
(b) There do not exist non-zero integers $a_0, b_0, \ldots, a_\ell, b_\ell$ such that $s_{2n} = \prod_{i=0}^\ell a_i^{(2n)^i}$ and $s_{2n-1} = \prod_{i=0}^\ell b_i^{(2n-1)^i}$ for all n.

We will give two different proofs of this result, one based on a theorem of Evertse (yielding, for a fixed finite set of primes $\mathcal{S}$, an effective bound on the number of non-degenerate solutions of an $\mathcal{S}$-unit equation in k variables over the rationals) and the other using only elementary methods.

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* Corresponding author.
E-mail addresses: leonetti.paolo@gmail.com (P. Leonetti), salvatore.tringali@uni-graz.at (S. Tringali).
URLs: https://sites.google.com/site/leonettipaolo/ (P. Leonetti), http://imsc.uni-graz.at/tringali (S. Tringali).

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As a corollary, we find that, for fixed $c_1, x_1, \ldots, c_k, x_k \in \mathbf{N}^+$, the number of distinct prime factors of $c_1 x_1^n + \cdots + c_k x_k^n$ is bounded, as n ranges over $\mathbf{N}^+$, if and only if $x_1 = \cdots = x_k$.

1. Introduction

Given $k, \ell \in \mathbf{N}^+$, let $x_{i,j}$ be, for $1 \le i \le k$ and $0 \le j \le \ell$, some fixed rationals. Then, consider the $\mathbf{Q}$-valued sequence $(s_n)_{n \ge 1}$ obtained by taking

$$s_n := \sum_{i=1}^{k} \prod_{j=0}^{\ell} x_{i,j}^{n^j} \tag{1}$$

for every $n \in \mathbf{N}^+$ (notations and terminology, if not explained, are standard or should be clear from the context); we refer to s_n as a sum of super-powers of degree ℓ. Notice that $(s_n)_{n \ge 1}$ includes as a special case any $\mathbf{Q}$-valued sequence of general term

$$\sum_{i=1}^{k} \prod_{j=1}^{\ell_i} y_{i,j}^{f_{i,j}(n)}, \tag{2}$$

where, for each $i = 1, \ldots, k$, we let $\ell_i \in \mathbf{N}^+$ and $y_{i,1}, \ldots, y_{1,\ell_i} \in \mathbf{Q} \setminus \{0\}$, while $f_{i,1}, \ldots, f_{i,\ell_i}$ are polynomials in one variable with integral coefficients. Conversely, sequences of the form (1) can be viewed as sequences of the form (2), the latter being prototypical of scenarios where polynomials are replaced with more general functions $\mathbf{N}^+ \to \mathbf{Z}$ (see also § 4).

We let $\omega(x)$ denote, for each non-zero $x \in \mathbf{Z}$, the number of distinct prime divisors of x, and define $\omega(0) := \infty$. Then, for $x \in \mathbf{Z}$ and $y \in \mathbf{N}^+$ we set $\omega(xy^{-1}) := \omega(\delta^{-1}x) + \omega(\delta^{-1}y)$, where δ is the greatest common divisor of x and y.

In addition, given $n \ge 2$ and $\theta > 1$, we write $\operatorname{slog}_\theta(n)$ for the super-logarithm of n to base θ, that is, the largest integer $\kappa \ge 0$ for which $\theta^{\otimes \kappa} \le n$, where $\theta^{\otimes 0} := 1$ and $\theta^{\otimes \kappa} := \theta^{\theta^{\otimes(\kappa-1)}}$ for $\kappa \ge 1$; note that $\operatorname{slog}_\theta(n) \to \infty$ as $n \to \infty$.

The main goal of this paper is to provide necessary and sufficient conditions for the boundedness of the sequence $(\omega(s_n))_{n \ge 1}$. More precisely, we have:

Theorem 1. *The following are equivalent:*

(a) *There is a base $\theta > 1$ such that $\omega(s_n) > \operatorname{slog}_\theta(n)$ for infinitely many n.*
(b) $\limsup_{n\to\infty} \omega(s_n) = \infty$.
(c) *There do not exist non-zero rationals $a_0, b_0, \ldots, a_\ell, b_\ell$ such that $s_{2n} = \prod_{j=0}^{\ell} a_j^{(2n)^j}$ and $s_{2n-1} = \prod_{j=0}^{\ell} b_j^{(2n-1)^j}$ for all n.*

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