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Period and index for higher genus curves

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Given a curve C over a field K, the period of C/K is the gcd of degrees of K-rational divisor classes, while the index is the gcd of degrees of K-rational divisors. S. Lichtenbaum showed that the period and index must satisfy certain divisibility conditions. For given admissible period, index, and genus, we show that there exists a curve C and a number field K with these desired invariants, as long as the index is not divisible by 4.

1. Period and index

Let K be a field; usually, we will consider it to be a number field. By a *curve* C over K, we shall mean a smooth projective geometrically integral curve. The *period* and *index* of C over K are two integer invariants which measure the failure of C to have rational points. Specifically, the index is the gcd of degrees of all effective divisors $D \in \operatorname{Div} C$—that is, effective divisors which are rational over K. Equivalently, the index is the gcd of degrees $[L : K]$, where L/K ranges over algebraic extensions such that $C(L) \neq \varnothing$. To see this, note that if $P \in C(L)$, then $D = \sum \sigma P$ is a rational effective divisor, where σ ranges over the embeddings of L into an algebraic closure of K; and

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conversely any minimal rational effective divisor is of this form. Also observe that if C already has K-points, then its index is 1.

The period is less stringent: here we look at the smallest positive degree of rational divisor *classes*; these are given by divisors which are linearly equivalent to their Galois conjugates. To see that the two invariants need not be the same, consider any conic over $\mathbb{R}$ without rational points, say the curve with affine piece $x^2 + y^2 = -1$. Certainly the index is 2, but as our curve is genus 0, there is a single divisor class of degree 1; namely, the class of a point. Therefore this class must be rational over $\mathbb{R}$, and so the period is 1.

As a rational divisor automatically belongs to a rational divisor class, we have $P \mid I$. Furthermore, the canonical class gives a rational divisor of degree $2(g-1)$, so $I \mid 2(g-1)$. Lichtenbaum in [Lic69] found further conditions on the possible values of the period and index:

Theorem 1.1 *(Theorem 8, [Lic69]). Let C/K be a curve over a field with genus g, period P, and index I. Then*

(i) $P \mid I \mid 2P^2$, and
(ii) if either $2(g-1)/I$ or P is even, then $I \mid P^2$.

We say a triple of integers (g, P, I) is *admissible* if they satisfy the divisibility conditions of Lichtenbaum's theorem; that is, if they are possible values for the genus, period, and index of a curve. The goal of this paper is to determine whether every admissible triple indeed occurs as the invariants of some curve over a number field. Our main result is the following:

Theorem 1.2. *Given any admissible triple (g, P, I) such that $4 \nmid I$, there exists a number field K and a curve Y over K with genus g, period P and index I.*

Similar results have been proven earlier, including a complete answer in the genus 1 case:

Theorem 1.3 *(Theorem 2, [Sha12]). Let $(1, P, I)$ be an admissible triple. Let E be an elliptic curve over a number field K. Then there exists a genus 1 curve X which is a principal homogeneous space for E with period P and index I.*

A slightly weaker version of the above was proved in [CS10], and was used to construct Shafarevich–Tate groups with arbitrarily high p-rank.

Over other fields there are a scattering of results. In the case of p-adic fields, Lichtenbaum found stricter divisibility conditions:

Theorem 1.4 *(Theorem 7, [Lic69]). If C is a curve over a finite extension of $\mathbb{Q}_p$ with genus g, period P, and index I, then $P \mid (g-1)$, $I \mid 2P$, and $I = P$ if $(g-1)/P$ is even.*

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