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Average values of L-functions in even characteristic [☆]

Sunghan Bae [a], Hwanyup Jung [b,*]

[a] Department of Mathematics, KAIST, Daejon 305-701, Republic of Korea
[b] Department of Mathematics Education, Chungbuk National University, Cheongju 361-763, Republic of Korea

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ABSTRACT

Let $k = \mathbb{F}_q(T)$ be the rational function field over a finite field $\mathbb{F}_q$, where q is a power of 2. In this paper we solve the problem of averaging the quadratic L-functions $L(s, \chi_u)$ over fundamental discriminants. Any separable quadratic extension K of k is of the form $K = k(x_u)$, where x_u is a zero of $X^2 + X + u = 0$ for some $u \in k$. We characterize the family $\mathcal{I}$ (resp. $\mathcal{F}$, $\mathcal{F}'$) of rational functions $u \in k$ such that any separable quadratic extension K of k in which the infinite prime $\infty = (1/T)$ of k ramifies (resp. splits, is inert) can be written as $K = k(x_u)$ with a unique $u \in \mathcal{I}$ (resp. $u \in \mathcal{F}$, $u \in \mathcal{F}'$). For almost all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \frac{1}{2}$, we obtain the asymptotic formulas for the summation of $L(s, \chi_u)$ over all $k(x_u)$ with $u \in \mathcal{I}$, all $k(x_u)$ with $u \in \mathcal{F}$ or all $k(x_u)$ with $u \in \mathcal{F}'$ of given genus. As applications, we obtain the asymptotic mean value formulas of L-functions at $s = \frac{1}{2}$ and $s = 1$ and the asymptotic mean value formulas of the class number h_u or the class number times regulator $h_u R_u$.

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* Corresponding author.
E-mail addresses: shbae@kaist.ac.kr (S. Bae), hyjung@chungbuk.ac.kr (H. Jung).

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1. Introduction

In Disquisitiones Arithmeticae [9], Gauss presented two famous conjectures concerning the average values of class numbers associated with binary quadratic forms over $\mathbb{Z}$, which can be restated as the average values of class numbers of orders in quadratic number fields. The imaginary case of these conjecture was first proved by Lipschitz, and the real case by Siegel [19]. By the Dirichlet's class number formula, these two conjectures can be stated as averages of the values of quadratic Dirichlet L-functions at $s = 1$. In [19], Siegel showed how to average over all discriminants. Let χ_d be the quadratic character defined by the Kronecker symbol $\chi_d(n) = (\frac{d}{n})$ and

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \chi_d(n) n^{-s}$$

be the Dirichlet series associated to χ_d. Siegel [19] has obtained averaging formulas for $L(1, \chi_d)$ over all positive discriminants d between 1 and N, or all negative discriminants d such that $1 < |d| \leq N$. From these averaging formulas with Dirichlet's class number formula, Siegel has obtained averaging formulas for the class number h_d or the class number times regulator $h_d R_d$ over all positive discriminants d between 1 and N, or all negative discriminants d such that $1 < |d| \leq N$. At the critical point $s = \frac{1}{2}$, Jutila [16] derived an asymptotic formula for

$$\sum_{d} L^k(\tfrac{1}{2}, \chi_d) \quad (k = 1, 2),$$

where d runs over fundamental discriminants in the interval $0 < d \leq X$. In [10], using the Eisenstein series of $\frac{1}{2}$-integral weight, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 1$, Goldfeld and Hoffstein obtained an asymptotic formula for $\sum L(s, \chi_m)$, where the sum is either over positive square-free m between 1 and N, or over negative square-free m with $1 < |m| \leq N$. Putting $s = 1$ and using the Dirichlet's class number formula, one can average the class number h_m or the class number times regulator $h_m R_m$ over the fields $\mathbb{Q}(\sqrt{m})$.

Now, we introduce the analogous results in function fields over finite fields. Let $k = \mathbb{F}_q(T)$ be the rational function field over a finite field $\mathbb{F}_q$ of q elements, and $\mathbb{A} = \mathbb{F}_q[T]$ be the ring of polynomials. First, we consider the case of q being odd. Let χ_N be the quadratic character defined by the Kronecker symbol $\chi_N(f) = (\frac{N}{f})$ and

$$L(s, \chi_N) = \sum_{\substack{f \in \mathbb{A} \\ f:\text{monic}}} \chi_N(f) |f|^{-s}$$

be the quadratic Dirichlet L-function associated to χ_N. In [12], Hoffstein and Rosen has obtained a result on averaging $L(1, \chi_N)$ over all non-square monic polynomials $N \in \mathbb{A}$ (all discriminants) of given degree. Using the averaging of $L(1, \chi_N)$ with

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