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The Legendre equation in Euclidean imaginary quadratic number fields

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A B S T R A C T

We show that the equation $x^2 + y^2 + z^2 = 0$, which has no nontrivial solutions in the ring of integers of $\mathbb{Q}(\sqrt{-7})$, does not have them in other quadratic number field extensions either. Then, we show that if the Legendre's equation with coefficients a, b, c in the ring of integers of $\mathbb{Q}(\sqrt{-d})$, for $d = 1, 2, 3, 7, 11$, has a solution (x, y, z), it has a solution with

a) $|z_0| \leq \sqrt{\frac{4}{3-d}|ab|}$, for $d = 1, 2$

b) $|z_0| \leq \sqrt{\frac{16d}{-d^2+14d-1}|ab|}$, for $d = 3, 7, 11$.

This represents an improvement, in the case $d = 1$, of bounds given previously.

1. Introduction

The Diophantine equation

$$ax^2 + by^2 + cz^2 = 0,$$

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(in its normal form), as shown by Legendre [5], has a nontrivial solution in integers if and only if the negative of the products in pairs of the coefficients are quadratic residues module the other coefficient and a, b, c do not have the same sign. Samet [7] showed that these same conditions are also necessary and sufficient to find nontrivial solutions, but now in the ring of the Gaussian integers. He also suggested how his method of proof might serve for other rings of integers of quadratic extensions of rational numbers. His method works on some of the cases suggested by him, but not for the suggested case $\mathbb{Q}(\sqrt{-7})$, as was shown by Hemer [1]. The equation $x^2+y^2+z^2=0$ satisfies these conditions but has no nontrivial solutions in the ring of integers of $\mathbb{Q}(\sqrt{-7})$. What motivates to ask when this equation in the ring of integers of other quadratic extensions is also not solvable. In four of the five imaginary Euclidean quadratic fields, $\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-11})$, there is a total analogy with the result of Legendre on the ring of rational integers. Hemer, using the index method, showed the following result:

If a, b and c are square-free integers, relatively prime in pairs and not zero, then the equation

$$ax^2 + by^2 + cz^2 = 0$$

has nontrivial integral solutions if and only if $-bc$, $-ca$, and$-ab$ are quadratic residues of a, b and c respectively. In the remaining field $\mathbb{Q}(\sqrt{-7})$, these conditions are also necessary but not sufficient, we must add the extra condition:

$$ax^2 + by^2 + cz^2 \equiv 0 \pmod{\pi^3},$$

where π is one of the prime divisors of 2, for instance $\frac{1+\sqrt{-7}}{2}$.

In 1950, Holzer [2] showed that if the equation $ax^2+by^2+cz^2=0$, in its normal form, is solvable in integers, then there exists a solution with

$$|x| \le \sqrt{|bc|}, |y| \le \sqrt{|ca|}, |z| \le \sqrt{|ab|}.$$

In this way, we can find a solution using these bounds. Later, Mordell [6] gave a simpler proof of Holzer's theorem.

Now, more recently, Leal-Ruperto [4] based on the proof given by Mordell, proved that Legendre's equation

$$ax^2 + by^2 + cz^2 = 0,$$

expressed in its normal form, when having a nontrivial solution in the Gaussian integers, has a solution (x, y, z) where $|z| \le \sqrt{(1+\sqrt{2})|ab|}$. In this paper, we give a modification of [4] which results in a similar version of the theorem, for Legendre's equation with coefficients a, b, c in the ring of integers of $\mathbb{Q}(\sqrt{-d})$, for $d = 1, 2, 3, 7, 11$. In the case of $d = 1$, we have improved the bound given by Leal-Ruperto.

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