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Some expressions for binary theta series by eta-quotients and their applications

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ABSTRACT

In this paper, we show that some single theta series of positive definite binary quadratic forms can be expressed in terms of eta-quotients. We also give two applications of our theorem as follows; (1) to prove that a certain modular form, which is a product of theta series and eta-product, has complex multiplication, (2) an explicit description for the image of $\eta(\tau)^2\eta(N\tau)^2$ under the Hecke operator T_2.

1. Introduction and notation

There are a lot of results which give expressions in terms of eta-quotients for the theta series associated to positive definite binary quadratic forms or their linear combinations, for example [1], [2]. In this paper, we prove that some "single" theta series of positive definite binary quadratic forms can be written as linear combinations of some eta-quotients. Also we give two applications of the main theorem. In the first application

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we show that a product of certain theta series and eta-product has complex multiplication by an imaginary quadratic field. This result is quite non-obvious, because it is not clear whether one can obtain a nice formula for the Fourier expansion of a product of two given modular forms. The second one gives an explicit formula for the image of weight two eta-product $\eta(\tau)^2\eta(N\tau)^2$ with $N \equiv -1 \pmod{24}$ under the Hecke operator T_2. It turns out that it is a product of certain theta series and eta-product of weight one.

Notation. Let N be a positive integer, k an integer and χ a Dirichlet character modulo N. Let $M_k(N, \chi)$ (resp. $S_k(N, \chi)$) denote the space of modular forms (resp. cusp forms) of weight k and level N with character χ. In the case where χ is the trivial character, we write $S_k(N, \chi)$ as $S_k(N)$. If $N \equiv -1 \pmod 4$, then let χ_N denote the Kronecker character $\left(\frac{-N}{\cdot}\right)$.

Let $\eta(\tau)$ denote the Dedekind eta-function;

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

with $q = e^{2\pi i\tau}, \tau \in \mathbb{C}$ and $\mathrm{Im}(\tau) > 0$. It is known that $\eta(\tau)$ has the theta series expression;

$$\eta(\tau) = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{\frac{n^2}{24}}. \tag{1}$$

2. Statement of the main result

Recall that if $ax^2 + bxy + cy^2$ is an integral positive definite quadratic form with discriminant $-N$, then the theta series $\sum_{x,y\in\mathbb{Z}} q^{ax^2+bxy+cy^2}$ belongs to $M_1(N, \chi_N)$.

Let a be a positive integer, and let N be a positive integer such that $N \equiv -1 \pmod{4a}$. Then the theta series

$$\Theta_a^N(\tau) := \sum_{x,y\in\mathbb{Z}} q^{ax^2+xy+\frac{N+1}{4a}y^2}$$

is an element of $M_1(N, \chi_N)$. Our main result is that for $a = 1, 2, 3, 4, 6$, the theta series $\Theta_a^N(\tau)$ can be expressed as a linear combination of some eta-quotients.

Theorem 2.1. *We have that*

(i) *if* $N \equiv -1 \pmod 4$, *then*

$$\Theta_1^N(\tau) = \frac{\eta(2\tau)^5\eta(2N\tau)^5}{\eta(\tau)^2\eta(4\tau)^2\eta(N\tau)^2\eta(4N\tau)^2} + 4 \cdot \frac{\eta(4\tau)^2\eta(4N\tau)^2}{\eta(2\tau)\eta(2N\tau)}, \tag{2}$$

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