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THE GEOMETRY OF NODAL SETS AND OUTLIER DETECTION

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ABSTRACT. Let (M,g) be a compact manifold and let $-\Delta\phi_k = \lambda_k\phi_k$ be the sequence of Laplacian eigenfunctions. We present a curious new phenomenon which, so far, we only managed to understand in a few highly specialized cases: the family of functions $f_N : M \to \mathbb{R}_{\geq 0}$

$$f_N(x) = \sum_{k \leq N} \frac{1}{\sqrt{\lambda_k}} \frac{|\phi_k(x)|}{\|\phi_k\|_{L^\infty(M)}}$$

and their extrema seem strangely suited for the detection of anomalous points on the manifold. It may be heuristically interpreted as the sum over distances to the nearest nodal line and potentially hints at a new phenomenon in spectral geometry. We give rigorous statements on the unit square $[0,1]^2$ (where minima localize in $\mathbb{Q}^2$) and on Paley graphs (where f_N recovers the geometry of quadratic residues of the underlying finite field $\mathbb{F}_p$). Numerical examples show that the phenomenon seems to arise on fairly generic manifolds.

1. INTRODUCTION.

1.1. **Introduction.** The purpose of this paper is to report a curious observation in spectral geometry that seems intrinsically interesting and may have nontrivial applications in outlier detection. Numerical examples on rough real-life data (see §3 below) indicate that the phenomenon is robust and seems to occur on fairly generic manifolds.

Observation. Let (M,g) be a compact manifold and let $-\Delta\phi_k = \lambda_k\phi_k$ denote the Laplacian eigenfunctions. The maxima and minima of the function

$$f_N(x) = \sum_{k \leq N} \frac{1}{\sqrt{\lambda_k}} \frac{|\phi_k(x)|}{\|\phi_k\|_{L^\infty(M)}}$$

seem to correspond to *special* points on the manifold.

The notion of *special* point is vague and depends on the context: the special points turn out to be the rational numbers on $[0,1]$, quadratic (non-)residues in finite fields $\mathbb{F}_p$ on Paley Graphs and sea-mines in sonar data. We have no theoretical understanding of the underlying phenomenon, nor do we understand its extent or the proper language in which it should be phrased.

1.2. **Number Theory on** $[0,1]$**.** A first indicator that this quantity may be of some interest was given by the third author [8] in the special case of the interval $[0,1]$.

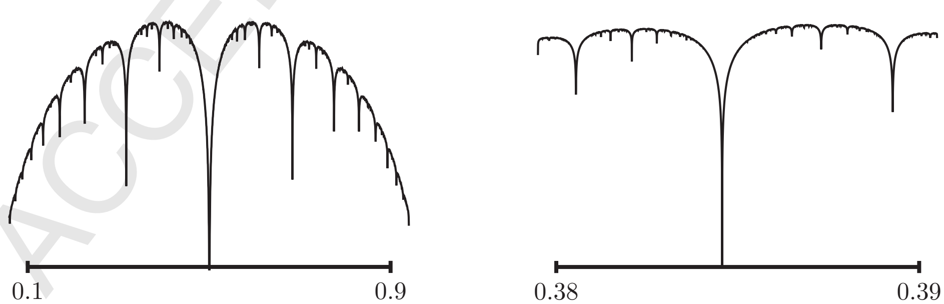

FIGURE 1. The function f_N for $N = 50000$ on $[0.1, 0.9]$ and zoomed in (right): local minima are located at rational numbers (the big cusp in the right is located at $x = 5/13$).

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