ARTICLE IN PRESS YJNTH:5863

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Slow continued fractions, transducers, and the Serret theorem

Giovanni Panti

Department of Mathematics, University of Udine, via delle Scienze 206, 33100 Udine, Italy

ARTICLE INFO

Article history:
Received 7 June 2017
Received in revised form 19 August 2017
Accepted 19 August 2017
Available online xxxx
Communicated by S.J. Miller

MSC:
11A55
37A45

Keywords:
Continued fractions
Gauss maps
Tail property
Extended modular group
Transducers

ABSTRACT

A basic result in the elementary theory of continued fractions says that two real numbers share the same tail in their continued fraction expansions iff they belong to the same orbit under the projective action of $\mathrm{PGL}_2\,\mathbb{Z}$. This result was first formulated in Serret's *Cours d'algèbre supérieure*, so we'll refer to it as to the Serret theorem.

Notwithstanding the abundance of continued fraction algorithms in the literature, a uniform treatment of the Serret result seems missing. In this paper we show that there are finitely many possibilities for the groups $\Sigma \leq \mathrm{PGL}_2\,\mathbb{Z}$ generated by the branches of the Gauss maps in a large family of algorithms, and that each Σ-equivalence class of reals is partitioned in finitely many tail-equivalence classes, whose number we bound. Our approach is through the finite-state transducers that relate Gauss maps to each other. They constitute opfibrations of the Schreier graphs of the groups, and their synchronizability—which may or may not hold—assures the a.e. validity of the Serret theorem.

E-mail address: giovanni.panti@uniud.it.

https://doi.org/10.1016/j.jnt.2017.08.034

Please cite this article in press as: G. Panti, Slow continued fractions, transducers, and the Serret theorem, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.08.034

1. Introduction

Let α, β be irrational numbers, with infinite regular continued fraction expansions $[0, a_1, a_2, \ldots]$, $[0, b_1, b_2, \ldots]$, respectively. It is a classical fact that these expansions have the same tail (i.e., there exists $t_1, t_2 \geq 0$ such that $a_{t_1+n} = b_{t_2+n}$ for every $n \geq 1$) if and only if α and β are conjugated by an element of the extended modular group $\mathrm{PGL}_2\,\mathbb{Z}$. This result first appeared as §16 of the third edition (1866) of Serret's *Cours d'algèbre supérieure* [22], the second edition (1854) making no mention of continued fractions; easily accessible modern references are [9, §10.11], [14, §9.6], [5, §2.7]. An equivalent reformulation is that α, β (without loss of generality in the real unit interval) are in the same $\mathrm{PGL}_2\,\mathbb{Z}$-orbit iff they have the same eventual orbit under the *Gauss map* $T : x \mapsto x^{-1} - \lfloor x^{-1} \rfloor$ (see Fig. 1 right). The key point here is that the cf. expansion of, say, α is nothing else than its T-symbolic orbit: $T^t(\alpha) \in \big[(a_{t+1}+1)^{-1}, a_{t+1}^{-1}\big)$ for every $t \geq 0$. We refer to [6, Chapter 7], [7, Chapter 3] and references therein for the interpretation of continued fractions in terms of dynamical systems.

Besides the regular "Floor" one, a great number of continued fraction algorithms appear in the literature, the complex of them forming a large passacaglia on the theme of the euclidean algorithm. As a definitely incomplete list we cite the Ceiling, Nearest Integer, Even, Odd, Farey fractions [2], the α-fractions [17], [1], and the (a, b)-fractions [13], not to mention algorithms with coefficients in rings of algebraic integers and multidimensional continued fractions. Asking for the status of the Serret result for these systems is then quite natural.

In this paper we give a fairly complete answer for the algorithms in a certain specific class, namely the class of accelerations of Gauss-type maps arising from finite unimodular partitions of a unimodular interval. After setting notation and stating a few well known facts, we introduce our class in §2; we provide various explicit examples, showing that our class, albeit nonexhaustive, contains many important and much studied algorithms. It is a fortunate fact that the validity—or lack of it—of the Serret property is untouched by the acceleration process, so that we can restrict to "slow" algorithms. In §3 we associate a graph $\mathcal{G}_T$ to each such algorithm T, and show that $\mathcal{G}_T$ is an opfibration of the Schreier graph of the group Σ_T generated by the branches of T. The rôle of Σ_T is clearly crucial; indeed, if α, β have the same eventual T-orbit then they must necessarily be Σ_T-equivalent. Thus, the key question becomes "In how many tail-equivalence classes is partitioned a given Σ_T-equivalence class?", the Serret property amounting to the constant answer "Precisely one". In §4 we show that the index of each Σ_T in $\mathrm{PGL}_2\,\mathbb{Z}$ is at most 8, so that there are finitely many possibilities for these groups. In §5 we introduce finite-state transducers, and employ them in two ways: in Lemma 5.1 to relate different algorithms to each other, and in Lemma 5.5 to compute the expansion of a rational function of α directly from the expansion of α; neither use is new, see [8, §3.5] for the first and [19], [15] for the second. In Theorem 5.3 we answer the question cited above: every Σ_T-equivalence class is partitioned in finitely many tail-equivalence classes, whose number is bounded by the *defect* of the algorithm. We also give an explicit cri-

Please cite this article in press as: G. Panti, Slow continued fractions, transducers, and the Serret theorem, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.08.034