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Euler sums and Stirling sums

Weiping Wang *, Yanhong Lyu

School of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China

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In this paper, using the Bell polynomials and the methods of generating function and integration, we establish various mixed Euler sums and Stirling sums, and present a unified approach to determining the evaluations of unknown Euler sums. As a result, we give the values of two Euler sums of weight 6, six sums of weight 7, twelve sums of weight 8, and twenty sums of weight 9. Thus, all the Euler sums of weights $\{6, 7, 9\}$ can be reduced to zeta values, and all the Euler sums of weight 8 can be reduced to linear sum $S_{2,6}$ and zeta values.

* Corresponding author.
E-mail addresses: wpingwang@yahoo.com, wpingwang@zstu.edu.cn (W. Wang).

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1. Introduction

The generalized *harmonic numbers* are defined by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} \quad \text{for } n, r = 1, 2, \ldots .$$

When $r = 1$, they reduce to the classical harmonic numbers, denoted as $H_n = H_n^{(1)}$. Harmonic numbers are important in various branches of combinatorics and number theory, and they also frequently appear in the analysis of algorithms and expressions of special functions.

In response to a letter from Goldbach in 1742, Euler considered sums of the form

$$S_{p,q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} ,$$

where p and q are positive integers with $q \geq 2$ (see Berndt [3, p. 253]). These sums are called the *linear Euler sums* today. Euler discovered that in the cases of $p = 1$, $p = q$, $p + q$ odd, $p + q$ even but with the pair (p, q) being restricted to the set $\{(2, 4), (4, 2)\}$, the linear sums have evaluations in terms of zeta values, i.e., the values of the Riemann zeta function $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$ at the positive integers. In particular, he proved that

$$S_{1,q} = \sum_{n=1}^{\infty} \frac{H_n}{n^q} = \left(1 + \frac{q}{2}\right) \zeta(q + 1) - \frac{1}{2} \sum_{k=1}^{q-2} \zeta(k + 1)\zeta(q - k) .$$

By specifying the parameter q, various special sums can be computed. For example, we have

$$S_{1,6} = \sum_{n=1}^{\infty} \frac{H_n}{n^6} = 4\zeta(7) - \zeta(3)\zeta(4) - \zeta(2)\zeta(5) .$$

Euler also extrapolated the general formula of $S_{p,q}$ for the case of $p + q$ odd, which was verified by Borwein et al. [5] in 1995 (see also [19, Theorem 3.1]). A typical example is

$$S_{3,4} = \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4} = 18\zeta(7) - 10\zeta(2)\zeta(5) .$$

Different from the linear sums, the *nonlinear Euler sums* involve products of at least two harmonic numbers. As showed by Flajolet and Salvy [19], the linear and nonlinear Euler sums can be defined in a unified way. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ be a partition of integer p into k summands, that is, $p = \pi_1 + \pi_2 + \cdots + \pi_k$ and $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_k$. The Euler sum of index (π, q) is

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