Journal of Number Theory 185 (2018) 194–217

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the Cesàro average of the numbers that can be written as sum of a prime and two squares of primes

Marco Cantarini

Università di Ferrara, Dipartimento di Matematica e Informatica, Via Machiavelli, 30, 44121 Ferrara, Italy

ARTICLE INFO

Article history:
Received 7 January 2017
Received in revised form 20 August 2017
Accepted 4 September 2017
Available online 16 October 2017
Communicated by the Principal Editors

MSC:
primary 11P32
secondary 44A10, 33C10

Keywords:
Goldbach-type theorems
Laplace transforms
Cesàro average

ABSTRACT

Let $\Lambda(n)$ be the Von Mangoldt function and $r_{SP}(n) = \sum_{m_1+m_2^2+m_3^2=n} \Lambda(m_1)\Lambda(m_2)\Lambda(m_3)$ be the counting function for the numbers that can be written as sum of a prime and two squares. Let N be a sufficiently large integer. We prove that

$$\sum_{n\leq N} r_{SP}(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{N^{k+2}\pi}{4\Gamma(k+3)} + E(N,k)$$

for $k > 3/2$, where $E(N,k)$ consists of lower order terms that are given in terms of k and sum over the non-trivial zeros of the Riemann zeta function.

1. Introduction

We continue the recent work of Languasco, Zaccagnini and the author on additive problems with prime summands. In [9] and [10] Languasco and Zaccagnini study the

E-mail address: cantarini_m@libero.it.

https://doi.org/10.1016/j.jnt.2017.09.001

Cesàro weighted explicit formula for the Goldbach numbers (the integers that can be written as sum of two primes) and for the Hardy–Littlewood numbers (the integers that can be written as sum of a prime and a square). Recently [2] the author wrote a paper regarding the Cesàro average of the integers that can be written as sum of a prime and two squares. In a similar manner, we will study a Cesàro weighted explicit formula for the integers that can be written as sum of a prime and two squares of primes. We will obtain an asymptotic formula with a main term and more terms depending explicitly on the zeros of the Riemann zeta function. This technique allow us to obtain a large number of terms in our asymptotic but unfortunately the bound $k > 3/2$ seems to be very difficult to improve. We recall that, for $k = 0$, the Cesàro weights vanish so a result for $k \geq 0$ would allow us to get an asymptotic for the mean of $r_{SP}(n)$.

We let

$$r_{SP}(n) = \sum_{m_1+m_2^2+m_3^2=n} \Lambda(m_1)\Lambda(m_2)\Lambda(m_3)$$

where $\Lambda(n)$ is the Von Mangoldt function and

$$M_1(N,k) = \frac{N^{k+2}\pi}{4\Gamma(k+3)} \tag{1}$$

$$M_2(N,k) = \frac{N^{k+1}\pi}{4}\sum_{\rho}\frac{N^{\rho}\Gamma(\rho)}{\Gamma(k+2+\rho)} - \frac{N^{k+3/2}\sqrt{\pi}}{2}\sum_{\rho}N^{\rho/2}\frac{\Gamma(\rho/2)}{\Gamma(k+5/2+\rho/2)} \tag{2}$$

$$\begin{aligned} M_3(N,k) =& \frac{N^{k+1/2}\sqrt{\pi}}{2}\sum_{\rho_1}\sum_{\rho_2}N^{\rho_1+\rho_2/2}\frac{\Gamma(\rho_1)\Gamma\left(\frac{\rho_2}{2}\right)}{\Gamma(k+3/2+\rho_1+\rho_2/2)} \\ &+\frac{N^{k+1}}{4}\sum_{\rho_1}\sum_{\rho_2}N^{\rho_1/2+\rho_2/2}\frac{\Gamma\left(\frac{\rho_1}{2}\right)\Gamma\left(\frac{\rho_2}{2}\right)}{\Gamma(k+2+\rho_1/2+\rho_2/2)} \end{aligned} \tag{3}$$

$$M_4(N,k) = \frac{N^k}{4}\sum_{\rho_1}\sum_{\rho_2}\sum_{\rho_3}\frac{N^{\rho_1+\rho_2/2+\rho_3/2}\Gamma(\rho_1)\Gamma\left(\frac{\rho_2}{2}\right)\Gamma\left(\frac{\rho_3}{2}\right)}{\Gamma(k+\rho_1+\rho_2/2+\rho_3/2)}. \tag{4}$$

The main result of this paper is the following

Theorem 1. *Let N be a sufficient large integer. We have*

$$\sum_{n\leq N} r_{SP}(n)\frac{(N-n)^k}{\Gamma(k+1)} = M_1(N,k)+M_2(N,k)+M_3(N,k)+M_4(N,k)+O\left(N^{k+1}\right)$$

for $k > 3/2$, where $\rho = \beta + i\gamma$, with or without subscripts, runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$.

Note that an upper bound for $M_i(N,k)$, $i = 2,\dots,4$ depends closely on β. Let us define