ARTICLE IN PRESS YJNTH:5873

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Quadratic forms and their Berggren trees

Byungchul Cha[a,*], Emily Nguyen, Brandon Tauber

[a] *Muhlenberg College, United States*

A R T I C L E   I N F O

Article history:
Received 20 February 2017
Received in revised form 13 September 2017
Accepted 14 September 2017
Available online xxxx
Communicated by S.J. Miller

Keywords:
Pythagorean triples
Quadratic forms
Continued fraction

A B S T R A C T

An old result of Berggren's says that there exist three 3×3 matrices N_1, N_2, N_3 with the following remarkable property: Start with $(3, 4, 5)$ or $(4, 3, 5)$ and multiply N_1, N_2, or N_3 by it in any order any number of times. This yields another primitive Pythagorean triple (x, y, z), that is, a triple of positive integers without common factor satisfying $x^2 + y^2 - z^2 = 0$. Furthermore, every primitive Pythagorean triple can be obtained uniquely this way. In other words, all primitive Pythagorean triples can be given a tree-like structure with each edge representing a multiplication by N_j. In this paper, we present a geometric algorithm for producing such trees that is applicable to any integral quadratic form. Although this algorithm does not always yield a tree, we find a few other trees arising from different quadratic forms.

1. Introduction

It is well-known that there are infinitely many *primitive Pythagorean triples*, that is, integer triples (x, y, z) satisfying $x^2 + y^2 - z^2 = 0$ such that $x, y, z > 0$ without common factor. Perhaps less known is the fact that all such triples can be given a certain tree-like structure via matrix multiplication as in Fig. 1. If we define

* Corresponding author.
E-mail address: cha@muhlenberg.edu (B. Cha).

https://doi.org/10.1016/j.jnt.2017.09.003
0022-314X/

Please cite this article in press as: B. Cha et al., Quadratic forms and their Berggren trees, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.09.003

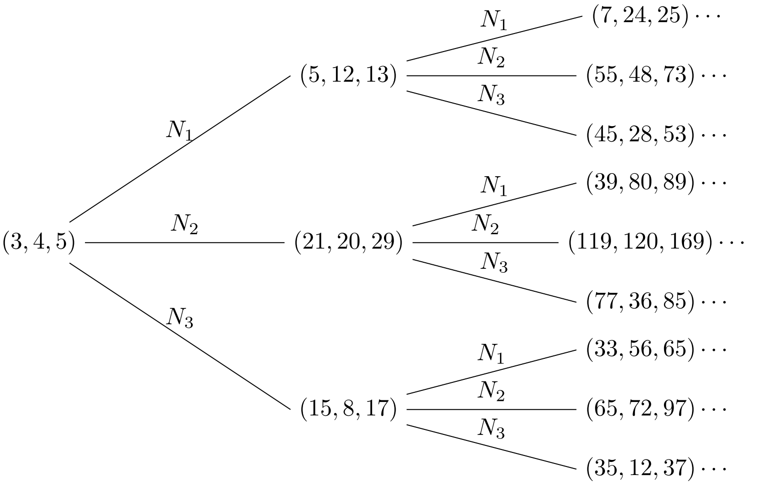

Fig. 1. The tree of Pythagorean triples.

$$N_1 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad N_3 = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix},$$

each (directed) edge in Fig. 1 between two adjacent Pythagorean triples represents a matrix multiplication, for example, $N_1(3,4,5)^\top = (5,12,13)^\top$. (Here, the "$\top$" means the transpose.) Then the theorem is that every primitive Pythagorean triple (x, y, z) (with y even) appears exactly once in the tree. If y is odd, (x, y, z) appears uniquely in a similar but different tree, having $(4, 3, 5)$ as its root with the edges generated by the same N_1, N_2, N_3 (see §A.1). As far as we are aware, [3] is the oldest literature containing this result. Afterwards, many authors seem to have (re-)discovered this theorem independently. See, for example, [1], [2], and other references cited in [9]. It turns out that the quadratic form $x^2 + y^2 - z^2$, which we will call the *Pythagorean form* from now on, is not the only one with such a property. In [10], Wayne found that a similar tree-like structure exists for triples satisfying $x^2 + xy + y^2 - z^2 = 0$.

Can the zeros of *other* quadratic forms be organized into trees this way? If such a tree exists, how can it be constructed? In this paper, we present some answers to these questions.

Let $Q(\mathbf{x})$ be an integral ternary quadratic form on $\mathbb{R}^3$. We will think of $Q(\mathbf{x})$ as a generalization of Pythagorean and Wayne's forms; thus, we will always assume that $Q(\mathbf{x})$ is nondegenerate and possesses infinitely many integer zeros (see (Q-I)—(Q-III) in §2.2 for precise conditions for $Q(\mathbf{x})$). The key idea is to construct the matrices such as N_1, N_2, N_3 using *reflections* of $\mathbb{R}^3$, whose definition is given in (5). This is done first by Conrad [5] in the case of Pythagorean triples.

In this paper, we attempt to generalize some of the ideas that surface in [5] and develop a geometric algorithm that can be applied to *any* integral quadratic form $Q(\mathbf{x})$

Please cite this article in press as: B. Cha et al., Quadratic forms and their Berggren trees, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.09.003