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On Diophantine exponents for Laurent series over a finite field

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A B S T R A C T

In this paper, we study the properties of Diophantine exponents w_n and w_n^* for Laurent series over a finite field. We prove that for an integer $n \geq 1$ and a rational number $w > 2n - 1$, there exist a strictly increasing sequence of positive integers $(k_j)_{j \geq 1}$ and a sequence of algebraic Laurent series $(\xi_j)_{j \geq 1}$ such that $\deg \xi_j = p^{k_j} + 1$ and

$$w_1(\xi_j) = w_1^*(\xi_j) = \ldots = w_n(\xi_j) = w_n^*(\xi_j) = w$$

for any $j \geq 1$. For each $n \geq 2$, we give explicit examples of Laurent series ξ for which $w_n(\xi)$ and $w_n^*(\xi)$ are different.

1. Introduction

Mahler [20] and Koksma [18] introduced Diophantine exponents which measure the quality of approximation to real numbers. Using the Diophantine exponents, they classified the set $\mathbb{R}$ all of real numbers. Let ξ be a real number and $n \geq 1$ be an integer. We denote by $w_n(\xi)$ the supremum of the real numbers w which satisfy

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$$0 < |P(\xi)| \leq H(P)^{-w}$$

for infinitely many integer polynomials $P(X)$ of degree at most n. Here, $H(P)$ is defined to be the maximum of the absolute values of the coefficients of $P(X)$. We denote by $w_n^*(\xi)$ the supremum of the real numbers w^* which satisfy

$$0 < |\xi - \alpha| \leq H(\alpha)^{-w^*-1}$$

for infinitely many algebraic numbers α of degree at most n. Here, $H(\alpha)$ is equal to $H(P)$, where $P(X)$ is the minimal polynomial of α over $\mathbb{Z}$.

We recall some results on Diophantine exponents. It is clear that $w_1(\xi) = w_1^*(\xi)$ for all real numbers ξ. Roth [29] established that $w_1(\xi) = w_1^*(\xi) = 1$ for all irrational algebraic real numbers ξ. Furthermore, it follows from the Schmidt Subspace Theorem that

$$w_n(\xi) = w_n^*(\xi) = \min\{n, d-1\} \tag{1}$$

for all $n \geq 1$ and algebraic real numbers ξ of degree d. It is known that

$$0 \leq w_n(\xi) - w_n^*(\xi) \leq n - 1$$

for all $n \geq 1$ and real numbers ξ (see Section 3.4 in [4]). Sprindžuk [32] proved that $w_n(\xi) = w_n^*(\xi) = n$ for all $n \geq 1$ and almost all real numbers ξ. Baker [3] proved that for $n \geq 2$, there exists a real number ξ for which $w_n(\xi)$ and $w_n^*(\xi)$ are different. More precisely, he proved that the set of all values taken by the function $w_n - w_n^*$ contains the set $[0, (n-1)/n]$ for $n \geq 2$. In recent years, this result has been improved. Bugeaud [10,5] showed that the set of all values taken by $w_2 - w_2^*$ is equal to the closed interval $[0, 1]$ and the set of all values taken by $w_3 - w_3^*$ contains the set $[0, 2)$. Bugeaud and Dujella [8] proved that for any $n \geq 4$, the set of all values taken by $w_n - w_n^*$ contains the set $\left[0, \frac{n}{2} + \frac{n-2}{4(n-1)}\right)$.

Let p be a prime. We can define Diophantine exponents w_n and w_n^* over the field $\mathbb{Q}_p$ of p-adic numbers in a similar way to the real case. Analogues of the above results for p-adic numbers have been studied (see e.g. Section 9.3 in [4] and [11,26]).

Let p be a prime and q be a power of p. Let us denote by $\mathbb{F}_q$ the finite field of q elements, $\mathbb{F}_q[T]$ the ring of all polynomials over $\mathbb{F}_q$, $\mathbb{F}_q(T)$ the field of all rational functions over $\mathbb{F}_q$, and $\mathbb{F}_q((T^{-1}))$ the field of all Laurent series over $\mathbb{F}_q$. For $\xi \in \mathbb{F}_q((T^{-1})) \setminus \{0\}$, we can write

$$\xi = \sum_{n=N}^{\infty} a_n T^{-n},$$

where $N \in \mathbb{Z}$, $a_n \in \mathbb{F}_q$ for all $n \geq N$, and $a_N \neq 0$. We define an absolute value on $\mathbb{F}_q((T^{-1}))$ by $|0| := 0$ and $|\xi| := q^{-N}$. This absolute value can be uniquely extended to the algebraic closure of $\mathbb{F}_q((T^{-1}))$ and we continue to write $|\cdot|$ for the extended absolute

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