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Selectivity in Definite Quaternion Algebras

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Abstract

We study the embedding problem for commutative orders into Eichler orders in definite quaternion algebras over the rationals by using methods from the theory of quadratic forms, specifically, Minkowski-Siegel's formula for the representation mass. We characterize when the Gaussian or Eisenstein integers embed into some but not all classes in the genus of an Eichler order. We also give an application to the theory of supersingular elliptic curves.

Keywords:
Selectivity, Quaternion Orders, Quaternion Algebras, Quadratic Forms, Local densities, Supersingular Elliptic Curves.

1. Introduction

A commutative order in a quaternion algebra B over a number field is called selective for the genus, $gen(\mathcal{D})$, of a full-rank order $\mathcal{D} \subset B$ if it embeds into some but not all orders in $gen(\mathcal{D})$. Selectivity might be seen as the lack of a Hasse principle for embedding and has applications in hyperbolic geometry as well as in the arithmetic of elliptic curves. In fact, the absence of selectivity for rank-2 commutative orders in the genus of maximal orders in indefinite quaternion algebras is key for the construction of isospectral-nonisometric hyperbolic surfaces and orbifolds (see [14], [15], or [22], p.129), while selectivity for rank-2 commutative orders in the genus of maximal orders in a definite quaternion algebra of prime discriminant relates to CM liftings of supersingular elliptic curves (see [10] and §5 below).

The tools that have been used to study selectivity in quaternion algebras and the applicability of this property to the aforementioned areas of mathematics, vary depending on the definiteness of the algebra at infinite places. The study of selectivity for commutative orders in indefinite quaternion algebras over number fields has been carried out by several authors (see [6] for $\mathcal{D}$ maximal, [4], [9] for $\mathcal{D}$ Eichler, and [13] for more general $\mathcal{D}$), this being a consequence of the spinor genera theory [1] (which establishes that under certain conditions the embedding problem can be studied using class field theory) for the group of automorphisms of B, since in this case (B indefinite), the set of spinor genera in the genus of an order $\mathcal{D}$ coincides with the set of classes in the genus of $\mathcal{D}$. Despite the fact that, for definite quaternion algebras, classes and spinor genera of orders are no longer in general the same, the theory of spinor exceptions [21] for representation of numbers by positive definite ternary quadratic forms might be read as selectivity for commutative orders into the genus of a full-rank order in a definite quaternion algebra. On the other hand, the genus of an Eichler order in a rational quaternion algebra is equal to its spinor