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Representations of Weil–Deligne groups and Frobenius conjugacy classes

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ABSTRACT

Let X be a smooth projective algebraic variety over a number field F, with an embedding $\tau : F \hookrightarrow \mathbb{C}$. The action of $\mathrm{Gal}(\bar{F}/F)$ on ℓ-adic cohomology groups $\mathrm{H}^i_{et}(X_{/\bar{F}}, \mathbb{Q}_\ell)$, induces Galois representations $\rho^i_\ell : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(\mathrm{H}^i_{et}(X_{/\bar{F}}, \mathbb{Q}_\ell))$. Fix a non-archimedean valuation v on F, of residual characteristic p. Let F_v be the completion of F at v and $'W_v$ be the Weil–Deligne group of F_v. We establish new cases, for which the linear representations $\underline{\rho^i_\ell}$ of $'W_v$, associated to ρ^i_ℓ, form a compatible system of representations of $'W_v$ defined over $\mathbb{Q}$. Under suitable hypotheses, we show that in some cases, these representations actually form a compatible system of representations of $'W_v$, with values in the Mumford–Tate group of $\mathrm{H}^i_B(\tau X(\mathbb{C}), \mathbb{Q})$. When X has good reduction at v, we establish a motivic relationship between the compatibility of the system $\{\rho^i_\ell\}_{\ell \neq p}$ and the conjugacy class of the crystalline Frobenius of the reduction of X at v.

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1. Introduction and notation

Throughout, F is a number field, with an embedding $\tau : F \hookrightarrow \mathbb{C}$, v is a non-archimedean valuation on F and F_v is the completion. By $\bar{F}$ we denote a fixed separable algebraic closure of F, $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ is an extension of τ, $\bar{v}$ is an extension of v to $\bar{F}$ and $\bar{F}_v$ is the localization of $\bar{F}$ at $\bar{v}$. The residue fields of F_v and $\bar{F}_v$ are denoted by k_v and $\bar{k}_v$, respectively. Let the characteristic of k_v be $p > 0$ and write $|k_v| := q_v$. We write $\Gamma_v := \mathrm{Gal}(\bar{F}_v/F_v) \subset \Gamma_F := \mathrm{Gal}(\bar{F}/F)$ and $I_v \subset \Gamma_v$ is the inertia group. By an *arithmetic Frobenius* $\Phi_v \in \Gamma_v$, we mean an element which induces the Frobenius automorphism ϕ_v of $\bar{k}_v$. We denote by W_v the *Weil group* of F_v, i.e., the dense subgroup of Γ_v which induce on $\bar{k}_v$ an integral power ${\phi_v}^{\alpha(w)}$, for $w \in \Gamma_v$. The map $\alpha : W_v \to \mathbb{Z}$ thus defined is a group homomorphism and $ker(\alpha) = I_v$. The *Weil–Deligne group* $'W_v$ of F_v is the group scheme over $\mathbb{Q}$ defined as the semi-direct product of W_v with the additive group $\mathbb{G}_a$ over $\mathbb{Q}$, on which W_v acts as: $w \cdot x \cdot w^{-1} = q_v^{\alpha(w)} \cdot x$. For ease of exposition, we shall assume our varieties to be geometrically irreducible.

Consider a smooth projective algebraic variety X over F. The action of Γ_F on the geometric ℓ-adic cohomology groups $V_\ell^i := H_{et}^i(X_{/\bar{F}}, \mathbb{Q}_\ell)$, induces Galois representations $\rho_\ell^i : \Gamma_F \to \mathrm{GL}(V_\ell^i)$. A fundamental problem in arithmetic geometry, is to determine, how far the properties of ρ_ℓ^i are independent of ℓ. For instance, it has been conjectured [28] that, *if v is any non-archimedean valuation on F, then for every $w \in W_v$, the characteristic polynomial $P_{\ell,v}^i(w,T) := \det(1 - \rho_\ell^i(w)T; V_\ell^i)$, of the $\mathbb{Q}_\ell$-linear map $\rho_\ell^i(w)$ has coefficients in $\mathbb{Q}$ and is independent of ℓ?* By Deligne's result [8] on the Weil conjectures, we know that this conjecture holds true if we assume that variety X has good reduction at v. But the case of bad-reduction is wide open. The starting point of this article is an observation (see Theorem 2.1), which gives a criterion for detecting the rationality and ℓ-independence of $P_{\ell,v}^i(w,T)$, irrespective of the type of reduction at v. This allows us to verify the above conjecture in a large number of new cases; see Corollary 2.4. Equipped with these results, we can take a step forward in the analysis of the bad reduction case. In order to do this, first one attaches to ρ_ℓ^i, a linear representation $\underline{\rho_\ell^i} := (V_\ell^i, {\rho^i}'_\ell, N'_{i,\ell})$ of $'W_v$ over $\mathbb{Q}_\ell$, where ${\rho^i}'_\ell$ is a continuous representation of W_v and $N'_{i,\ell}$ is the associated monodromy operator (see §3.2). Then, we investigate the following conjecture:

Conjecture 1.1 *($C_{WD}(X_{F_v}, i)$, cf. [12, 2.4.3]). The system $\{\underline{\rho_\ell^i}\}_{\ell \neq p}$ forms a compatible system (in the sense of [7, 8.8]) of linear representations of $'W_v$ defined over $\mathbb{Q}$.*

The notion of compatible system considered here is a strong one (see §3.2). The first main result of this article establishes some new cases of the above conjecture.

Theorem 1.2. *Let X a smooth projective variety over F which is a finite product of moduli spaces of stable vector bundles of co-prime rank and degree over smooth projective curves, unirational varieties of dimension ≤ 3, uniruled surfaces, hyperkähler varieties of $K3^{[n]}$*

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