
# Small prime solutions of a nonlinear equation

Zhixin Liu

PII: S0022-314X(17)30355-4
DOI: https://doi.org/10.1016/j.jnt.2017.09.014
Reference: YJNTH 5884

To appear in: *Journal of Number Theory*

Received date: 18 April 2017
Accepted date: 3 September 2017

# SMALL PRIME SOLUTIONS OF A NONLINEAR EQUATION

ZHIXIN LIU

ABSTRACT. Let $a_1, \cdots, a_4$ be non-zero integers and $n$ any integer. Suppose that $a_1, \cdots, a_4$ and $n$ satisfy some related conditions. In this paper we prove that
(i) if $a_j$ are not all of the same sign, then the equation $a_1p_1 + a_2p_2^2 + a_3p_3^2 + a_4p_4^2 = n$ has prime solutions satisfying $\max\{p_1, p_2^2, p_3^2, p_4^2\} \ll |n| + \max\{|a_j|\}^{14+\varepsilon}$;
(ii) if all $a_j$ are positive and $n \gg \max\{|a_j|\}^{15+\varepsilon}$, then the equation $a_1p_1 + a_2p_2^2 + a_3p_3^2 + a_4p_4^2 = n$ is soluble in primes $p_j$.

## 1. Introduction

Let $n$ be an integer, and let $a_1, \cdots, a_4$ be non-zero integers. We consider here the nonlinear equation in the form

$$a_1p_1 + a_2p_2^2 + a_3p_3^2 + a_4p_4^2 = n, \tag{1.1}$$

where $p_j$ are prime variables. We shall assume the condition of the congruent solubility for (1.1), that is

$$N(q) \geq 1 \qquad \text{for all } q \geq 1, \tag{1.2}$$

where

$$N(q) := \text{card}\{(n_1, \cdots, n_4) : 1 \leq n_j \leq q, (n_j, q) = 1, \\ a_1n_1 + a_2n_2^2 + a_3n_3^2 + a_4n_4^2 \equiv n (\text{mod}\, q\}.$$

We also suppose that

$$(a_i, a_j) = 1, \quad 1 \leq i < j \leq 4, \tag{1.3}$$

and write $A = \max\{2, |a_1|, \cdots, |a_4|\}$. The main results in this paper are the following two parallel theorems.

**Theorem 1.1.** *Suppose (1.2) and (1.3). If $a_1, \cdots, a_4$ are not all of the same sign, then (1.1) has solutions in primes $p_j$ satisfying*

$$\max\{p_1, p_2^2, p_3^2, p_4^2\} \ll |n| + A^{14+\varepsilon},$$

*where the implied constant depends only on $\varepsilon$.*

**Theorem 1.2.** *Suppose (1.2) and (1.3). If $a_1, \cdots, a_4$ are all positive, then (1.1) is soluble whenever*

$$n \gg A^{15+\varepsilon}, \tag{1.4}$$

*where the implied constant depends only on $\varepsilon$.*

2010 *Mathematics Subject Classification.* 11P32, 11P05, 11P55.
*Key words and phrases.* small prime, Waring-Goldbach problem, circle method.